A Gentle Introduction to Representation Stability

" Stability Properties of Moduli Spaces " — R. J. Rolland, J. C. H. Wilson

(I) (Co)homology of a Group

G - group

Fact : $\exists$ a topological space, called the $K(G,1)$ - space X,

s.t.

$$\pi_1(X) \cong G$$

and $\begin{matrix} \tilde{X} \\ \downarrow \\ X \end{matrix}$ univ. cover contractible

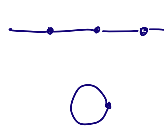
X is unique upto homotopy

Defn / Fact :

The (co)homologies of this $K(G,1)$ space X are the (co)homologies of the group G.

Examples

1) $G = \mathbb{Z}$

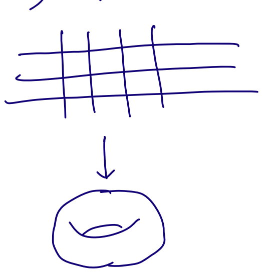


$\mathbb{R}$

$\downarrow$

S^1 — $K(\mathbb{Z},1)$ - space

2) $G = \mathbb{Z} \times \mathbb{Z}$



$\mathbb{R}^2$

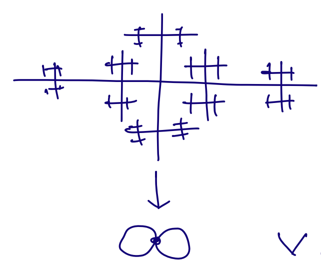
$\downarrow$

$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$

$- K(\mathbb{Z}^2, 1)$

3) $G = F_n$

For F_2 :

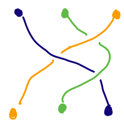


$\bigvee_n S^1$ — $K(F_n, 1)$

Example 4. (Switch to side board)

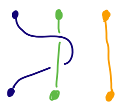
The braid group B_n.

B_n = group of braids on n-strands.



$B_n \twoheadrightarrow S_n$

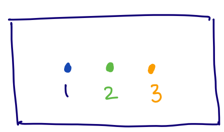
$\ker(B_n \twoheadrightarrow S_n) =: PB_n$ ("pure braids")



$1 \to PB_n \to B_n \to S_n \to 1$

Braids as fundamental groups:

$PConf_n(\mathbb{C}) := \{(x_1, \dots, x_n) \mid x_i \in \mathbb{C} \text{ distinct}\}$



$\pi_1(PConf_n\, \mathbb{C}) = PB_n$

$Conf_n(\mathbb{C}) := \{(x_1, \dots, x_n)\} / \sim$ (permuting the particles)

$\pi_1(Conf_n\, \mathbb{C}) = B_n$

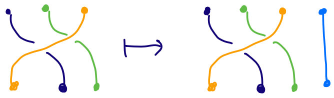
Fact: $Conf_n\, \mathbb{C}$, $PConf_n\, \mathbb{C}$ are in fact the $K(G,1)$'s for B_n, PB_n.

So we can study the (co)homologies of braids by studying these configuration spaces.

II Homological Stability

Eg: B_n: family of groups indexed by n grow in complexity

$i_n : B_n \hookrightarrow B_{n+1}$



$$(i_n)_* : H_k(B_n) \longrightarrow H_k(B_{n+1})$$

$$(i_n)_* : H_1(B_n) \longrightarrow H_1(B_{n+1})$$

Fact: $H_1(B_n) = B_n^{ab} \cong \mathbb{Z}$ for $n \geq 2$

and

$(i_n)_* : H_1(B_n) \to H_1(B_{n+1})$ is an iso for $n \geq 2$

$H_2(B_n) \cong \mathbb{Z}_2$ for $n \geq 4$

and

$(i_n)_* : H_2(B_n) \longrightarrow H_2(B_{n+1})$ is an iso for $n \geq 4$.

In general,

$(i_n)_* : H_k(B_n) \longrightarrow H_k(B_{n+1})$ is an iso for $n \geq 2k$.

So even though the groups are growing, in some sense their dim k structure (eventually) stabilizes.
This is called "homological stability".

Other families of groups that satisfy homological stability:

$\{S_n\}_n$, $\{MCG(\Sigma_g)\}_g$, $\{GL_n(R)\}_n$

B_n^{ab}:

$B_n = \{\sigma_1, \dots, \sigma_n)$ $\quad \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$

$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

In the abelianization:

$\sigma_i^2\sigma_{i+1} = \sigma_i\sigma_{i+1}^2$

$\sigma_i = \sigma_{i+1}$

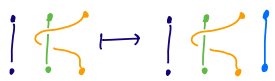
$B_n^{ab} = \langle \bar{\sigma}_1 \rangle \cong \mathbb{Z}$

III Representation Stability

What happens when we don't have homological stability?

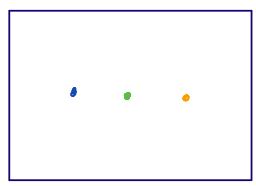
<u>Example</u>: PB_n

$i_n : PB_n \hookrightarrow PB_{n+1}$



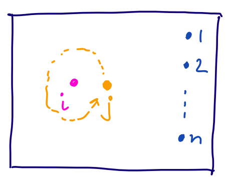
What's $H_1(PB_n)$?

$\rightarrow$ View homology classes as "particle dances"



$H_1(PB_n) \cong \mathbb{Z}^{\binom{n}{2}}$

Generated by α_{ij}, $i<j$:



<u>Not</u> homologically stable

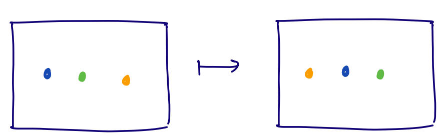
But they "stabilise as S_n-representations".

The rest of this talk will be about making this precise.

We'll move to homology with $\mathbb{Q}$-coefficients.

$H_k(PB_n; \mathbb{Q})$ is a $\mathbb{Q}$-vector space.

$S_n \curvearrowright PConf_n(\mathbb{C})$



$S_n \curvearrowright H_k(PB_n; \mathbb{Q})$

So the $\mathbb{Q}$-vector space $H_k(PB_n; \mathbb{Q})$ can be viewed as an S_n-representation.

$$\cdots \; H_1(PB_4;\mathbb{Q}) \xrightarrow{(i_4)_*} H_1(PB_5;\mathbb{Q}) \xrightarrow{(i_5)_*} H_1(PB_6;\mathbb{Q}) \xrightarrow{(i_6)_*} H_1(PB_7;\mathbb{Q}) \xrightarrow{(i_7)_*} \cdots$$

$$\circlearrowright S_4 \qquad \circlearrowright S_5 \qquad \circlearrowright S_6 \qquad \circlearrowright S_7$$

$$V_{(4)} \oplus V_{(3,1)} \oplus V_{(2,2)} \qquad V_{(5)} \oplus V_{(4,1)} \oplus V_{(3,2)} \qquad V_{(6)} \oplus V_{(5,1)} \oplus V_{(4,2)}$$

Fact :
- Any S_n-rep can be uniquely written as a direct sum of irreducibles.
- The irreducibles of S_n are in canonical bijection with partitions of n.

Eg : $(3,1)$ is a partition of 4

Denote the corresponding irrep as $V_{(3,1)}$

Remarks :
- The underlying reason behind this stability can be understood through FI-modules (Modules over the category FI)

  Thm : For a finitely generated FI-module, we have S_n-representation stability in the sense described above.

- We have similar stability results for $PConf_n(M)$ (resp; $Conf_n(M)$) (instead of $PConf_n(\mathbb{C})$ (resp; $Conf_n(\mathbb{C})$)).

- Other families of groups besides S_n also arise naturally when studying rep. stability of different families of spaces.

(If time left, sketch out
1. FI-Modules
2. Viewing $PConf_n$ / $H_1(PConf_n)$ as an FI-Module
3. Finite generation)

IV FI-modules

The FI-category: Category of finite sets and injective maps

$$
\begin{array}{ccccccccccccccc}
[1] & \xhookrightarrow{i_1} & [2] & \xhookrightarrow{i_2} & [3] & \xhookrightarrow{i_3} & [4] & \xhookrightarrow{i_4} & [5] & \xhookrightarrow{i_5} & \cdots & \xhookrightarrow{i_{n-1}} & [n] & \xhookrightarrow{i_n} & \cdots \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft & & \\
S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & & & S_n & &
\end{array}
$$

An FI-module is a functor V from the FI-category to the category of R-modules, for some commutative ring R.

Let's use V_n to denote the image of the object $[n]$ under this functor

An FI-module:

$$
\begin{array}{ccccccccccccccc}
V_1 & \xrightarrow{V(i_1)} & V_2 & \xrightarrow{V(i_2)} & V_3 & \xrightarrow{V(i_3)} & V_4 & \xrightarrow{V(i_4)} & V_5 & \xrightarrow{V(i_5)} & \cdots & \xrightarrow{V(i_{n-1})} & V_n & \xrightarrow{V(i_n)} & \cdots \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft & & \\
S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & & & S_n & &
\end{array}
$$

$H_1(PConf_n\, \mathbb{C})$ as an FI module:

$$
\begin{array}{cccccccc}
H_1(PB_1; \mathbb{Q}) & \xrightarrow{(i_1)_*} & H_1(PB_2; \mathbb{Q}) & \xrightarrow{(i_2)_*} & H_1(PB_3; \mathbb{Q}) & \xrightarrow{(i_3)_*} & H_1(PB_4; \mathbb{Q}) & \longrightarrow \cdots \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & \\
S_1 & & S_2 & & S_3 & & S_4 & \\
\alpha & & \alpha_{12} & & \alpha_{12} & & \alpha_{12} \quad \alpha_{14} & \\
 & & & & \alpha_{23} & & \alpha_{23} \quad \alpha_{24} & \\
 & & & & \alpha_{13} & & \alpha_{13} \quad \alpha_{34} &
\end{array}
$$

Note that :
- In a given $H_1(PB_n; \mathbb{Q})$, and $\sigma \in S_n$, we have $\sigma \cdot (\alpha_{12}) = \alpha_{\sigma(1)\sigma(2)}$.
  Thus we can obtain all the generators from α_{12}
- This is a specific case of a finitely generated FI-module
  (here the FI-module is finitely generated by $\{\alpha, \alpha_{12}\}$
- A general theorem states that for a finitely generated FI-module, we have representation stability, in the sense described earlier