Day 1 : The (Co)homology of $\mathrm{Conf}_n \mathbb{R}^2$

Configuration Spaces

M : d-dim manifold

- $\mathrm{Conf}_n(M) := \{(x_1, x_2, \ldots, x_n) \mid x_i \in M,\ x_i \neq x_j\}$ } "Ordered Configurations"

  $\downarrow$ dim: dn $\quad\subset\ M^n$

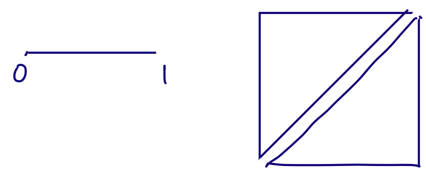
Eg:

Two points in $\mathrm{Conf}_n \mathbb{R}^3$

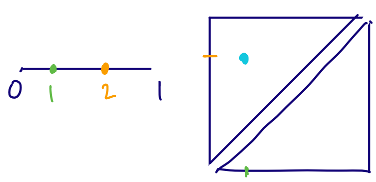
- $\mathrm{UConf}_n(M) := \mathrm{Conf}_n M / S_n$ } "Unordered Configurations"

  $\hookrightarrow$ acts by permuting labels

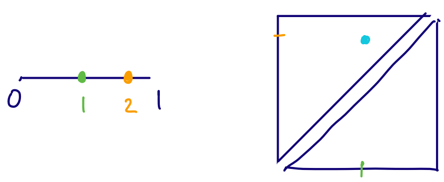
Example :
- $\mathrm{Conf}_1(M) \cong M$
- $\mathrm{Conf}_2([0,1]) \cong [0,1]^2 \setminus \{x=y\}$



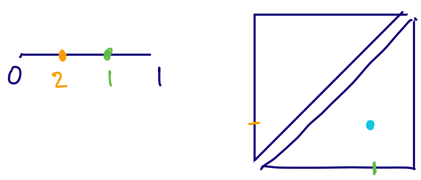
$\mathrm{Conf}_2([0,1])$



$\mathrm{Conf}_2([0,1])$



$\mathrm{Conf}_2([0,1])$



$\mathrm{Conf}_2([0,1])$

In most cases, we can't visualise $\mathrm{Conf}_n M$ (if $n>1$ and $\dim M > 1$, $\mathrm{Conf}_n M$ becomes ≥ 4 dim)

So to understand the space, a first step is to understand its homology.

We shall study the (co)homology of $\mathrm{Conf}_n \mathbb{R}^2$.

Why this topic?

→ $H_*(\mathrm{Conf}_n \mathbb{R}^d)$ has a very pretty combinatorial description; we'll get to see applications of useful geometric topology techniques in computing it.

→ Good prototypical example to understand homological & representation stability.

→ $H_*(\mathrm{Conf}_n \mathbb{R}^2)$ is precisely (pure) braid group homology.
$H_*(\mathrm{Conf}_n \mathbb{R}^\infty)$ is the homology of S_n.
Generalisations: RAAGs, MCGs,
Hyperplane Complements

→ Little Disks Operad, E_k-algebras

Minicourse Outline

Day 1 : Combinatorial Description of $H_*(\mathrm{Conf}_n \mathbb{R}^2)$ and $H^*(\mathrm{Conf}_n \mathbb{R}^2)$

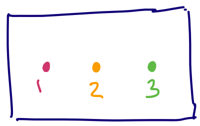
Day 2 : Some Proofs

Day 3 : Introduction to Homological Stability

Day 4 : Scanning Argument to calculate stable homology

Day 5 : Little Disks Operad, E_k-algebras

I $H_*(\mathrm{Conf}_n(\mathbb{R}^2))$ and Trees



A point in $\mathrm{Conf}_3 \mathbb{R}^2$

- $H_1(\mathrm{Conf}_n(\mathbb{R}^2)) \rightsquigarrow$ "1-dim structure"
  view loops as "particle dances"

Eg: 

View this as a map

$$S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$$

So get induced map

$$\underset{\mathbb{Z}}{\underset{\text{\rotatebox{90}{$\cong$}}}{H_1(S^1)}} \longrightarrow H_1(\mathrm{Conf}_n \mathbb{R}^2)$$

We can use this strategy to study higher-degree H_k.

- $H_2(\mathrm{Conf}_n \mathbb{R}^2) \rightsquigarrow$ "2-dim structure"

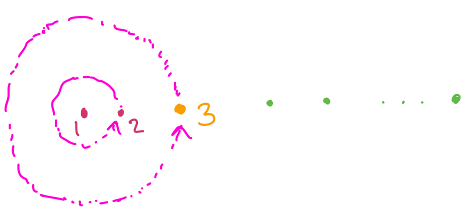
  We can construct maps

$$S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$$

And look at the (images of) induced map

$$\underset{\mathbb{Z}}{\underset{\text{\rotatebox{90}{$\cong$}}}{H_2(S^1 \times S^1)}} \longrightarrow H_2(\mathrm{Conf}_n \mathbb{R}^2)$$

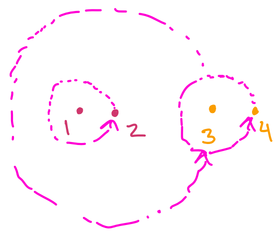
Eg: 

Eg: 

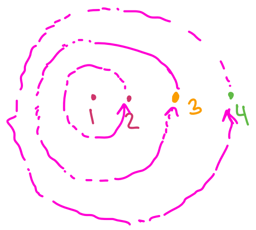
• $H_3(\mathrm{Conf}_n \mathbb{R}^2)$

$S^1 \times S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$

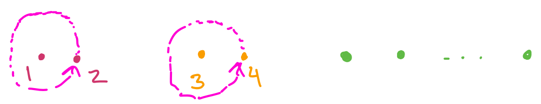
Eg:

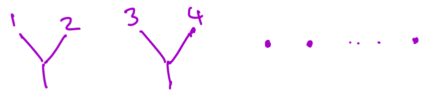


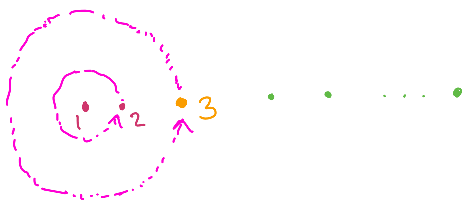
Eg:

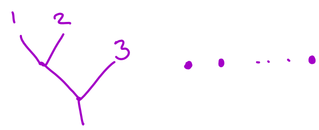


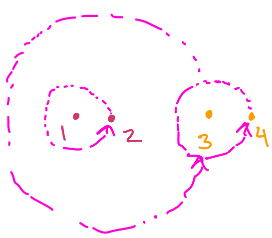
So we're constructing homology classes via "orbiting planet systems".
We can represent these orbiting systems using trees:

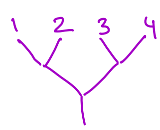


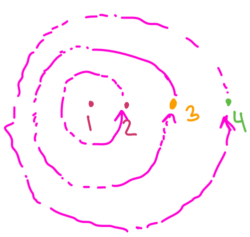


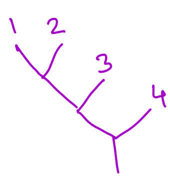


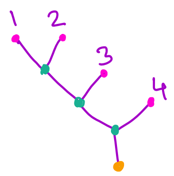










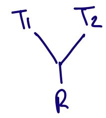


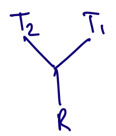
- internal vertices $\rightsquigarrow$ # internal vertices $|T|$ gives the homology degree
- root vertex
- leaves $\rightsquigarrow$ # leaves gives the no. of particles.

Note: • We have so far only described some homology gp. elements. We can get others by, for eg, taking linear combinations of them.

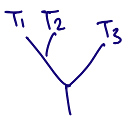
• Turns out, the above described classes generate all the homology. But they have some relations between them, i.e. they are not a basis.

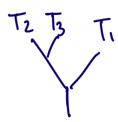
Relations:

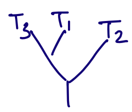
• 

$-\ (-1)^{|T_1||T_2|}$ 

$= 0$ (Anti-Symmetry)

• 

$+$ 

$+$ 

$= 0$ (Jacobi)

(II) Cohomology & Graphs

How do we start thinking about H^*?

Just for the purposes of this talk, $H^k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{Z})$

(In general, we always have a natural surjection $H^k \to \mathrm{Hom}(H_k, \mathbb{Z})$, which is an iso if, for eg, all H_k's are torsion-free)

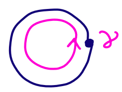
So for eg. $H^1(S^1) \cong \mathrm{Hom}(\underset{\substack{\text{\rotatebox{90}{$\cong$}}\\ \mathbb{Z}}}{H_1(S^1)}, \mathbb{Z}) \cong \mathbb{Z}$

To study $H^1(\mathrm{Conf}_n \mathbb{R}^2)$, we'll construct maps

$$\mathrm{Conf}_n \mathbb{R}^2 \longrightarrow S^1$$

to get induced

$$\mathbb{Z} \cong H^1(S^1) \to H^1(\mathrm{Conf}_n \mathbb{R}^2)$$

- $\omega \in H^1(S^1)$ 

S^1

γ : generates $H_1(S^1)$

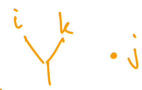
Have $\omega \in H^1(S^1)$ sending $\gamma \mapsto 1$

Analogous cohomology class in $H^1(\mathrm{Conf}_n \mathbb{R}^2)$:

$\alpha_{ij} \in H^1(\mathrm{Conf}_n \mathbb{R}^2)$

α_{ij} "extracts the motion of i wrt j"

$\alpha_{ij}($  $) = 1$

 $\alpha_{ij}($  $) = 0$

Rigorously, we have a map

$$a_{ij} : \mathrm{Conf}_n \mathbb{R}^2 \longrightarrow S^1$$

$$(x_1, \ldots, x_n) \mapsto \frac{x_i - x_j}{|x_i - x_j|}$$

ω : Generator of

$$\boxed{\alpha_{ij} := a_{ij}^*(\omega)}$$

Represent α_{ij} by a (directed) graph 

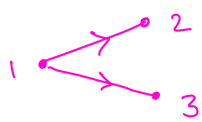
So,

- we've described some elements of $H^1(\mathrm{Conf}_n \mathbb{R}^n)$
- turns out, they also generate H^1
- we understand their action on H_1, as $H_1 \longrightarrow \mathbb{Z}$

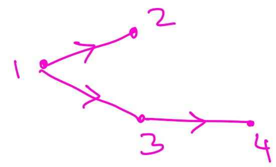
$\rightarrow$ What about higher H^k?

Using the cohomology cup product, we can get some higher-degree cohomology classes.

We'll represent these as graphs too.

Eg: $\alpha_{12}\alpha_{13} \in H^2$ 

$\alpha_{34}\alpha_{12}\alpha_{13} \in H^3$

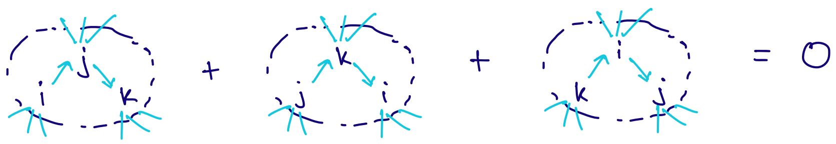


→ Implicit in these graphs is an ordering σ of the edges, to record the order of multiplication of the α_{ij}'s.

Note: # edges $|E(G)|$ ⇝ degree of cohomology $|E(G)|$

Relations

- $\alpha_{ij}\alpha_{jk} + \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij} = 0$ (Arnold)



- Changing the ordering σ of the edges $E(G)$ induces a corresponding sign change.

  And $i \to j = i \leftarrow j$

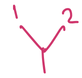
→ But how can we understand these graphs as $\mathrm{Hom}(H_k, \mathbb{Z})$?

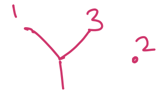
III Graph-Tree Pairing

So far, we have a way of associating cohomology classes to directed labelled graphs (with an ordering on edges)

To understand how they act on H_k, we need to unpack how the cup product works.

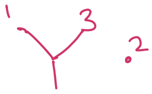
But it turns out, there is a combinatorial rule that captures this.

This should, for eg, give us: $\langle 1 \to 2,$  $\rangle = 1$

$\langle 1 \to 2,$  $\rangle = 0$

Here's how we define the general pairing $\langle G, T \rangle$:

1. If there's an edge $i \to j$ in G s.t. there is no path b/w i & j in T, then $\underline{\langle G, T \rangle = 0}$

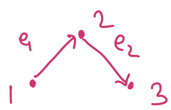
   Eg : $\langle 1 \to 2,$  $\rangle = 0$

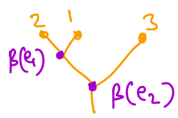
2. Otherwise, define

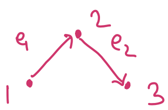
   $\beta_{G,T} : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$

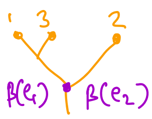
   $i \to j \quad \mapsto \quad$ lowest vertex of path from i to j.

Eg:









$$\langle G, T \rangle = \begin{cases} 0 & \text{if } \beta_{G,T} \text{ not a bijection} \\ \pm 1 & \text{o/w} \end{cases}$$

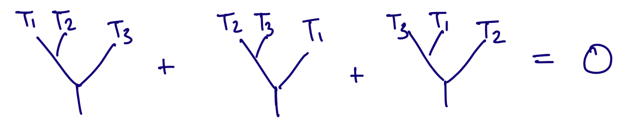
depends on σ and T

Day 2 : Some Proofs

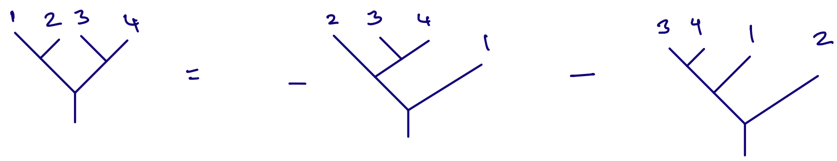
Goal : Show that the H_* and H^* corresponding to the trees & graphs from Day 1 generate all the (co)homology of $\mathrm{Conf}_n \mathbb{R}^2$

(I) Paring down to "talls trees" and "long lines"

Recall the Jacobi identity:

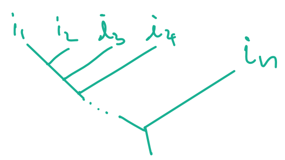


This implies, for eg, that



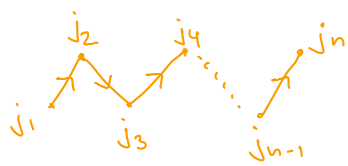
Note how both terms on the right hand side have exactly one vertex to the right of the lowest trivalent vertex.

In general, we can use the Jacobi and Anti-Symmetry relations to show that the H_* generated by binary forests is in fact generated by those whose components are all "tall trees"



A tall tree
(here i_1 is minimal among the i_k)

Similarly, we can use the Arnold identity to show that the H^* generated by graphs is in fact generated by those graphs whose components are all "long graphs".



A long graph
(here j_1 is minimal among the j_k)

→ The graph-tree pairing on tall trees and long graphs is perfect, in the sense that it is 1 iff $(i_1, i_2, \dots, i_n) = (j_1, \dots, j_n)$, and 0 otherwise.

This lets us deduce linear independence of these H_* and H^* elements, and thus gives us a lower bound on the ranks of H_* and H^*.

(II) The Forget-A-Point Fibration

Now that we have a lower bound on the ranks of H_* and H^*, we'll find an upper bound. If this upper bound agrees with the lower bound, this will show that all the H_* and H^* is generated by graphs and trees.

We have a surjective map

$$\mathrm{Conf}_n \mathbb{R}^2 \xrightarrow[\text{}]{\text{forget } n\text{th point}} \mathrm{Conf}_{n-1} \mathbb{R}^2$$

This is a fibration, and the fibers are $\mathbb{R}^2 - \{n-1 \text{ pts}\} \simeq \bigvee_{n-1} S^1$

So we get

$$\begin{array}{ccc} \bigvee_{n-1} S^1 & & \\ \downarrow & & \\ \mathrm{Conf}_n \mathbb{R}^2 & \xrightarrow{\text{forget } n\text{th point}} & \mathrm{Conf}_{n-1} \mathbb{R}^2 \end{array}$$

Fibrations are a topological analogue of short exact sequences. Analogous to how, for a s.e.s. of finitely generated abelian groups

$$0 \to K \to M \to N \to 0$$

we have $\mathrm{rk}(M) \leq \mathrm{rk}(K \oplus N)$,

for the above fibration we have

$$\mathrm{rk}\, H_k(\mathrm{Conf}_n \mathbb{R}^2) \leq \mathrm{rk}\, H_k\Big(\bigvee_{n-1} S^1 \times \mathrm{Conf}_{n-1} \mathbb{R}^2\Big)$$
$$= \bigoplus_{0 \leq i \leq k} \mathrm{rk}\Big(H_i\big(\bigvee_{n-1} S^1\big) \otimes H_{k-i}(\mathrm{Conf}_{n-1} \mathbb{R}^2)\Big)$$

(in fact, it turns out we have equality above).

Thus we get an upper bound on $H_*(\mathrm{Conf}_n \mathbb{R}^2)$ in terms of $H_*(\bigvee_{n-1} S^1)$ and $H_*(\mathrm{Conf}_{n-1} \mathbb{R}^2)$.

We can then get an upper bound on $H_*(\mathrm{Conf}_{n-1} \mathbb{R}^2)$, and so on, using our "tower of fibrations"

$$\begin{tikzcd}
\bigvee_{n-1} S^1 \arrow[r] & \mathrm{Conf}_n \mathbb{R}^2 \arrow[d] \\
& \vdots \arrow[d] \\
\bigvee_2 S^1 \arrow[r] & \mathrm{Conf}_3 \mathbb{R}^2 \arrow[d] \\
S^1 \arrow[r] & \mathrm{Conf}_2 \mathbb{R}^2 \arrow[d] \\
& \mathrm{Conf}_1 \mathbb{R}^2 \cong \mathbb{R}^2
\end{tikzcd}$$

And it turns out these upper bounds agree with the lower bounds given by tall trees and long lines.

III Proving the Arnold Relation

We want to show

$$\alpha_{ij}\alpha_{jk} + \alpha_{jk}\alpha_{ki} + \alpha_{ki}\alpha_{ij} = 0$$

Assume WLOG that $(i, j, k) = (1, 2, 3)$

Pf Idea : To express this as a cup product that is demonstably 0. We'll use the fact that cup products are dual to intersections.

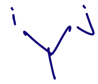
Proof : Since $\mathrm{Conf}_3 \mathbb{R}^d$ is a manifold, its cohomology is Poincare-Lefschetz dual to its locally finite homology.
Consider the submanifold of $(x_1, x_2, x_3) \in \mathrm{Conf}_3 \mathbb{R}^d$ s.t. x_1, x_2, x_3 are collinear. This submanifold has

3 components. Let Col_i denote the one in which x_i is in the middle.

Col_i is a submanifold of codimension $d-1$

(Suppose $x_i = (x_i^1, \ldots, x_i^d)$. Then x_1, x_2, x_3 being collinear means that $\dfrac{x_1^j - x_2^j}{x_1^j - x_3^j} = \dfrac{x_1^1 - x_2^1}{x_1^1 - x_3^1}$ for all $2 \le j \le d$. This gives us $d-1$ constraints)

When oriented, Col_i gives a locally finite homology class in codim $d-1$.

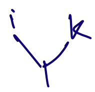
Thus the Poincare-Lefschetz dual of Col_i is in $H^{d-1}(\mathrm{Conf}_3 \mathbb{R}^d)$. Thus it is a linear combination of $\alpha_{12}, \alpha_{23}, \alpha_{31}$. We can find what this combination is by intersecting Col_i with various 

.

(*)

(see explanation for why this is true at the end of the proof)

Note: Col_i can only intersect 

and 

and does so at exactly one point each.

Moreover, these intersections differ in sign by -1, coming from orientation-reversing of the line on which the 3 points lie.

Thus the dual of Col_i is $\pm(\alpha_{ij} - \alpha_{ik})$.

Since Col_1 and Col_2 are disjoint, their duals cup product to 0.

Thus: $0 = (\alpha_{12} - \alpha_{13})(\alpha_{23} - \alpha_{21})$

$= \alpha_{12}\alpha_{23} - \alpha_{12}\alpha_{21} - \alpha_{13}\alpha_{23} + \alpha_{13}\alpha_{21}$

$= \alpha_{12}\alpha_{23} - 0 - (-1)^{d+(d-1)}\alpha_{23}\alpha_{31} + (-1)^{2d}\alpha_{31}\alpha_{12}$

$= \alpha_{12}\alpha_{23} + \alpha_{23}\alpha_{31} + \alpha_{31}\alpha_{12}$

Explanation of (*)

Here's a general fact:

Let M be an m-dim manifold, and $\omega \in H^k(M)$.

Let $\sigma \in H_k(M)$ and let $\omega_L \in H_{m-k}(M)$ be the Poincare-Lefschetz dual of ω.

Then,

$$\omega(\sigma) = (\text{sign of}) \; \sigma \cap \omega_L$$

(In our case, $\omega = Col_i$, $\sigma =$ 

)

<u>Here's why:</u> Let $[M]$ denote the fundamental class of M.
Let $\sigma_L \in H^{m-k}(M)$ be the dual of σ.

Then,

$$\omega(\sigma) = \omega([M] \frown \sigma_L) \overset{\substack{\text{cap-cup} \\ \text{relation}}}{=\!=} (\sigma_L \smile \omega)([M])$$

$$\overset{\substack{\text{cup-intersection} \\ \text{duality}}}{=\!=} (\text{sgn}) \; \sigma \cap \omega_L$$

Day 3 : Homological Stability

(I) Group (Co)homology

G - discrete group

Fact : $\exists$ a topological space X, called the classifying space BG s.t.

$$\pi_1(X) \cong G$$

and $\tilde{X} \to X$ univ. cover contractible

X is unique upto homotopy

Eg (1) $G = \mathbb{Z}$; $BG = S^1$
(2) $G = \mathbb{Z}^2$; $BG = S^1 \times S^1$
(3) $G = F_2$; $BG = S^1 \vee S^1$

(II) Homological Stability

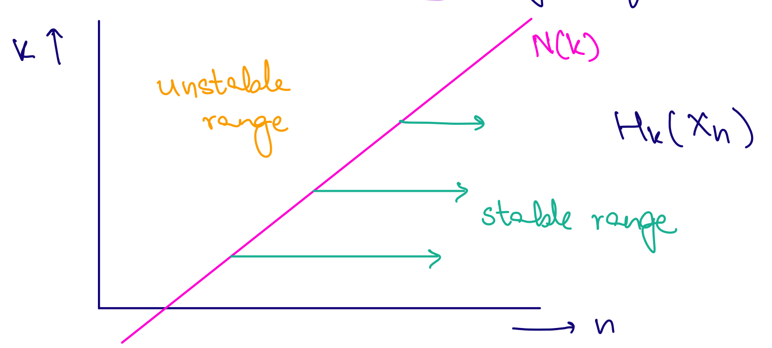
Suppose we have a sequence of spaces or groups $\{X_n\}$ w/ natural inclusions

$$i_n : X_n \longrightarrow X_{n+1} \qquad \Big\} \text{ "spaces are growing"}$$

If, for fixed k, $\exists N(k)$ s.t.

$$H_k(X_n) \xrightarrow[\cong]{i_*} H_k(X_{n+1}) \quad \forall\, n \geq N(k) \Big\} \text{ "k-dim structure stabilises"}$$

Then $\{X_n\}$ is said to be homologically stable

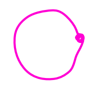


Homological stability arises in many naturally occurring families of spaces.

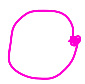
Eg: $\mathbb{RP}^n$

$\mathbb{RP}^0$

e^0

$\mathbb{RP}^1$ 

$e^0 \cup e^1 \cup e^2$

$\mathbb{RP}^2$  + 2-cell attached via $z \mapsto z^2$

$e^0 \cup e^1 \cup e^2$

In general, $\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$

Chain complex: $\mathbb{Z} \to \dots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

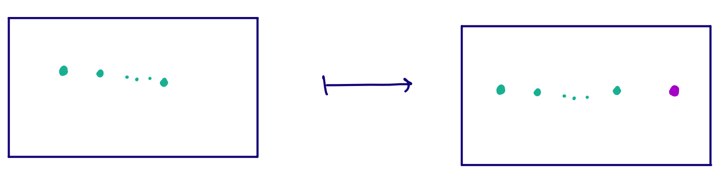
Attaching an $(n+1)$-cell does not affect H_* in deg $\leq n-1$.

So, in general, $H_k(\mathbb{RP}^n) \xrightarrow{\cong} H_k(\mathbb{RP}^{n+1})$ is an iso for $n \geq k+2$.

Eg: $\mathrm{UConf}_n \mathbb{R}^2$

We need inclusion maps $\mathrm{UConf}_n \mathbb{R}^2 \longrightarrow \mathrm{UConf}_{n+1} \mathbb{R}^2$

Need a way to "continuously add a new point"

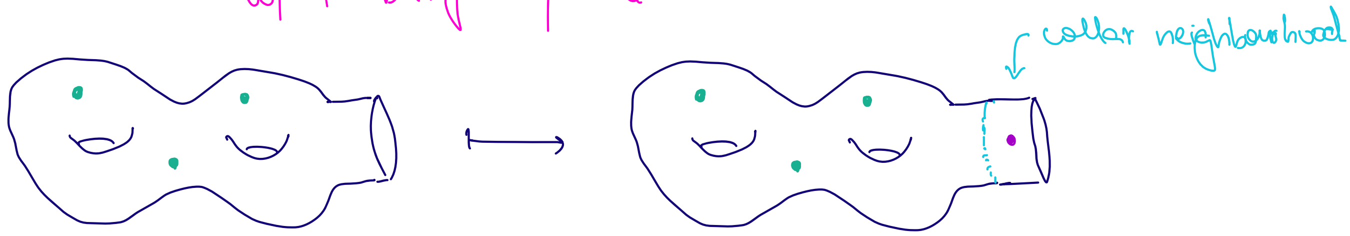


One way : $(z_1, \dots, z_n) \longmapsto (z_1, \dots, z_n, |z_1| + |z_2| + \dots + |z_n| + 1)$

These maps induce isos on H_k for $n \geq 2k$

Eg: $\mathrm{UConf}_n \underbrace{\Sigma_{g,1}}$

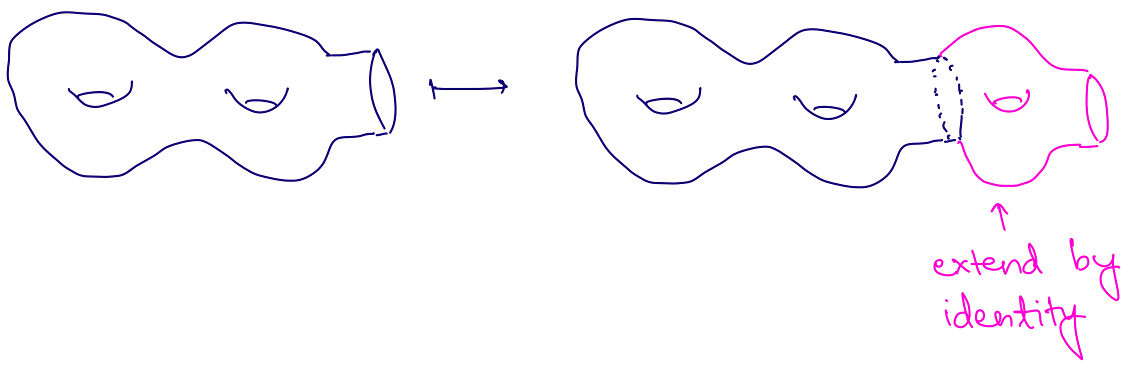
genus g surface
w/ 1 bdry component



Eg: Mapping Class Groups

$$\mathrm{Mod}(\Sigma_{g,1}) := \frac{\mathrm{Diffeo}(\Sigma_{g,1}, \partial \Sigma_{g,1})}{\text{Isotopy}}$$

$$\mathrm{Mod}(\Sigma_{g,1}) \longrightarrow \mathrm{Mod}(\Sigma_{g+1,1})$$



III Why homological stability?

In the example of $\mathbb{RP}^n$, we computed H_* to prove hom. stability.

But a strength of hom. stability is that we can often prove homological stability ranges, without knowing what the H_* groups are.

This can cut down the work of computing H_*, as it then suffices to only find H_* in the unstable range.

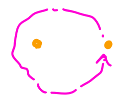
Even more remarkably, there are ways to compute the stable homology, without even computing any unstable H_*.

<u>Thm [Madsen-Weiss, 2007]</u> : Computed the stable H_* of $\mathrm{Mod}(\Sigma_{g,1})$

(IV) Stability in $H_*(\mathrm{UConf}_n \mathbb{R}^2)$

Let's think about $H_1(\mathrm{UConf}_n \mathbb{R}^2)$:

We have stability!

- $H_1(\mathrm{UConf}_1 \mathbb{R}^2) \cong H_1(\mathbb{R}^2) \cong 0$
- $H_1(\mathrm{UConf}_2 \mathbb{R}^2) \cong \mathbb{Z}$, generated by 
- $H_1(\mathrm{UConf}_3 \mathbb{R}^2) \cong \mathbb{Z}$, generated by 
- $\vdots$

Now let's think about $H_2(\mathrm{Conf}_n \mathbb{R}^2)$.

H_2 is generated by all possible orbit systems with 2 orbits, so let's first list all such orbits.

Note that the most no. of particles you can have in such an orbit system is <u>4</u>. This suggests we might start seeing hom. stability at $n=4$...

Indeed, we have:

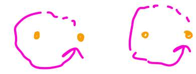
$$H_2(\mathrm{UConf}_1 \mathbb{R}^2) \cong H_2(\mathbb{R}^2) \cong 0$$

$$H_2(\mathrm{UConf}_2 \mathbb{R}^2) \cong 0$$ (can't make 2 orbits with only 2 particles)

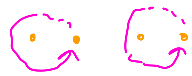
$H_2(UConf_3\,\mathbb{R}^2) \cong 0$ (Turns out, in homology,

$\beta =$  becomes 0, since both $2\beta = 0$ and $3\beta = 0$)

Stability!

$H_2(UConf_4\,\mathbb{R}^2) \cong \mathbb{Z}_2$, generated by 

(Turns out, in homology,

$\alpha =$  is equal to its negative, and so $2\alpha = 0$)

$H_2(UConf_5\,\mathbb{R}^2) \cong \mathbb{Z}_2$, generated by 

$\vdots$

> In general, in deg k, we have stability for $n \geq 2k$.
> i.e. $H_k(UConf_n\,\mathbb{R}^2) \underset{i_*}{\cong} H_k(UConf_{n+1}\,\mathbb{R}^2)$ for $n \geq 2k$.

V $Conf_n\,\mathbb{R}^2$ and Representation stability

Recall: $H_1(Conf_n\,\mathbb{R}^2)$ is freely generated by 

for all possible pairs (i,j).

Thus, $H_1(Conf_n\,\mathbb{R}^2) \cong \mathbb{Z}^{\binom{n}{2}}$ $\longrightarrow$ not stable!

The problem here arises because now we care about labels.
All the homology classes have the same "shapes" as before —
but now adding a particle gives us many new ways to label our points.
Luckily, there is a framework under which we can express this as a form of stability...

Let γ_{ij} denote the loop 

Thus $H_1(Conf_n\,\mathbb{R}^2) \cong \bigoplus_{1 \leq i < j \leq n} \mathbb{Z}\gamma_{ij} \cong \mathbb{Z}^{\binom{n}{2}}$

Note that $S_n \curvearrowright \mathrm{Conf}_n \mathbb{R}^2$ by permuting labels, and so

$S_n \curvearrowright H_1(\mathrm{Conf}_n \mathbb{R}^2)$

We have the following diagram:

$$
\begin{array}{ccccccccc}
H_1(\mathrm{Conf}_1 \mathbb{R}^2) & \xrightarrow{(i_1)_*} & H_1(\mathrm{Conf}_2 \mathbb{R}^2) & \xrightarrow{(i_2)_*} & H_1(\mathrm{Conf}_3 \mathbb{R}^2) & \xrightarrow{(i_3)_*} & H_1(\mathrm{Conf}_4 \mathbb{R}^2) & \longrightarrow & \cdots \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\
S_1 & & S_2 & & S_3 & & S_4 & & \\
0 & & \gamma_{12} & & \gamma_{12} & & \gamma_{12} \quad \gamma_{14} & & \\
 & & & & \gamma_{23} & & \gamma_{23} \quad \gamma_{24} & & \\
 & & & & \gamma_{13} & & \gamma_{13} \quad \gamma_{34} & &
\end{array}
$$

<u>Note that</u> :

① $\underset{\in H_1(\mathrm{Conf}_n)}{(i_n)_*(\gamma_{ij})} = \gamma_{ij} \in H_1(\mathrm{Conf}_{n+1})$

② For $\sigma \in S_n$, $\sigma \cdot \gamma_{ij} = \gamma_{\sigma(i)\sigma(j)}$

<u>Thus</u> : Notice that we can "obtain all possible $\gamma_{ij} \in H_1(\mathrm{Conf}_n \mathbb{R}^2)$" from $\gamma_{12} \in H_1(\mathrm{Conf}_2 \mathbb{R}^2)$ and the S_n-action.

The above picture is an example of a <u>finitely generated</u> <u>FI - module</u> .

The <u>FI - category</u> : Category of finite sets and injective maps

$$
\begin{array}{ccccccccccccccc}
[1] & \xhookrightarrow{i_1} & [2] & \xhookrightarrow{i_2} & [3] & \xhookrightarrow{i_3} & [4] & \xhookrightarrow{i_4} & [5] & \xhookrightarrow{i_5} & \cdots & \xhookrightarrow{i_{n-1}} & [n] & \xhookrightarrow{i_n} & \cdots \\
\circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \circlearrowright & & & & \circlearrowright & & \\
S_1 & & S_2 & & S_3 & & S_4 & & S_5 & & & & S_n & &
\end{array}
$$

An <u>FI-module</u> is a functor V from the FI-category to the category of R-modules, for some commutative ring R.

Let's use V_n to denote the image of the object $[n]$ under this functor

An FI-module:

$$\begin{tikzcd} V_1 \arrow[r, "V(i_1)"] \arrow[loop below, "S_1"] & V_2 \arrow[r, "V(i_2)"] \arrow[loop below, "S_2"] & V_3 \arrow[r, "V(i_3)"] \arrow[loop below, "S_3"] & V_4 \arrow[r, "V(i_4)"] \arrow[loop below, "S_4"] & V_5 \arrow[r, "V(i_5)"] \arrow[loop below, "S_5"] & \cdots \arrow[r, "V(i_{n-1})"] & V_n \arrow[r, "V(i_n)"] \arrow[loop below, "S_n"] & \cdots \end{tikzcd}$$

→ said to be <u>finitely generated</u> if $\exists\, d \in \mathbb{N}$ s.t. $\forall\, n > d$,

V_n can be "obtained from $V_1, \ldots, V_d$ by the arrows in the picture".

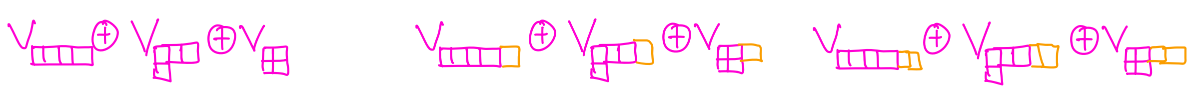
Thus $\{H_1(\mathrm{Conf}_n \mathbb{R}^2)\}$ is a fin. gen. FI-module.

This phenomenon is called "<u>representation stability</u>" (The groups are "stabilising as S_n-representations")

<u>Thm [Church - Ellenberg - Farb]</u>

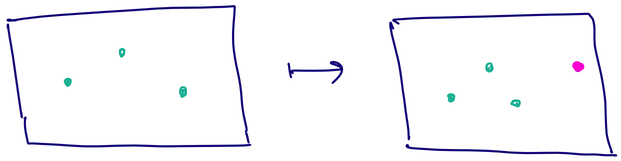
For a finitely generated FI-module, we have S_n-representation stability in the sense illustrated below:

$$\begin{tikzcd} \cdots & H_1(\mathrm{Conf}_4 \mathbb{R}^2; \mathbb{Q}) \arrow[r, "(i_4)_*"] \arrow[loop below, "S_4"] & H_1(\mathrm{Conf}_5 \mathbb{R}^2; \mathbb{Q}) \arrow[r, "(i_5)_*"] \arrow[loop below, "S_5"] & H_1(\mathrm{Conf}_6 \mathbb{R}^2; \mathbb{Q}) \arrow[r, "(i_6)_*"] \arrow[loop below, "S_6"] & H_1(\mathrm{Conf}_7 \mathbb{R}^2; \mathbb{Q}) \arrow[r, "(i_7)_*"] \arrow[loop below, "S_7"] & \cdots \end{tikzcd}$$



Day 4 : Scanning Arguments

Last Time : $\mathrm{UConf}_n \mathbb{R}^2 \longrightarrow \mathrm{UConf}_{n+1} \mathbb{R}^2$



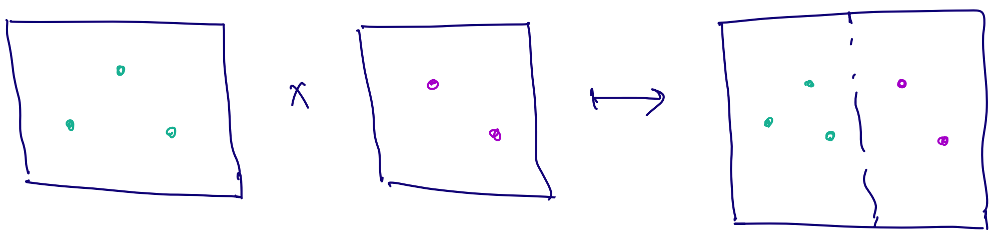
This induces maps on H_k

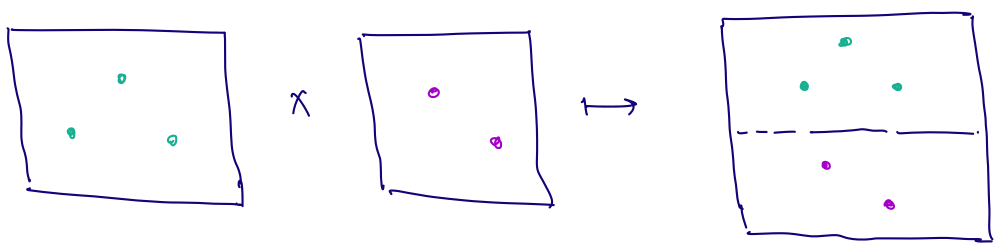
For fixed k, when $n \geq 2k$, we have isos

$$H_k(\mathrm{UConf}_n \mathbb{R}^2) \xrightarrow{\cong} H_k(\mathrm{UConf}_{n+1} \mathbb{R}^2)$$

Q : How do we calculate the stable H_* ?

$\coprod_n \mathrm{UConf}_n \mathbb{R}^2$ has two monoidal structures; i.e. by stacking configurations horizontally and vertically.





These sorts of monoidal structures make for a good setting for scanning arguments.

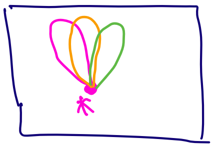
Here's the main goal for today :

Thm : The stable H_* of $\mathrm{UConf}_n \mathbb{R}^2$ is $\cong H_*(\Omega^2 S^2)$
The stable H_* of $\mathrm{UConf}_n \mathbb{R}^d$ is $\cong H_*(\Omega^d S^d)$

(I) Loop Spaces

X : topological space, basept $*$

ΩX : $(S^1, *) \to (X, *)$, compact-open topology



X

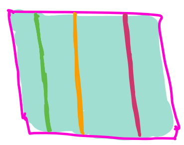
Path in $\Omega X \leftrightarrow$ htpy of (based) loops in X

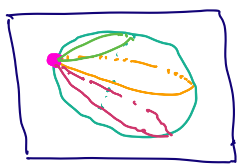
Path components of $\Omega X \leftrightarrow \pi_1 X$

$\Omega^n X := \Omega(\Omega^{n-1} X)$

Equivalent to: $(S^n, *) \to (X, *)$

$S^2 = I^2 / \partial I^2$





X

(II) Constructing BG

G : discrete group

BG : $\pi_1 = G$

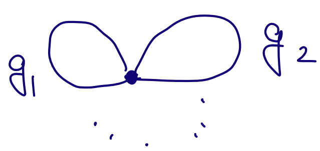
univ. cover $\simeq *$ ($\Leftrightarrow \pi_k = 0$ for $k \geq 2$)

Eg: $G = \mathbb{Z}$, $BG = S^1$; $G = \mathbb{Z} \times \mathbb{Z}$, $BG = S^1 \times S^1$

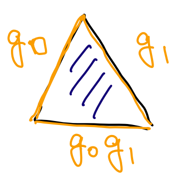
The "bar construction" for constructing a Δ-complex model of BG:

single vertex •

edges / loops for $g \in G$



2-simplices $g_0 | g_1$

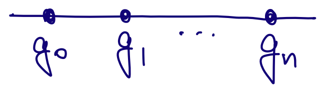


⋮

$(n+1)$-simplices $\qquad g_0 | g_1 | \cdots | g_n$

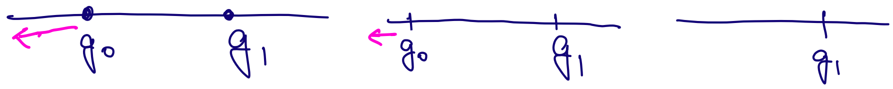
$$\partial = g_1 | \cdots | g_n + \Sigma (-1)^i g_0 | g_1 | \cdots | g_i g_{i+1} | \cdots | g_n + (-1)^n g_0 | g_1 | \cdots | g_n$$

Another model

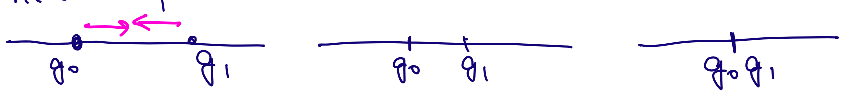


Points are allowed to "fall off the ends" or collide (and labels get multiplied)

i.e. the following depicts a path in this space



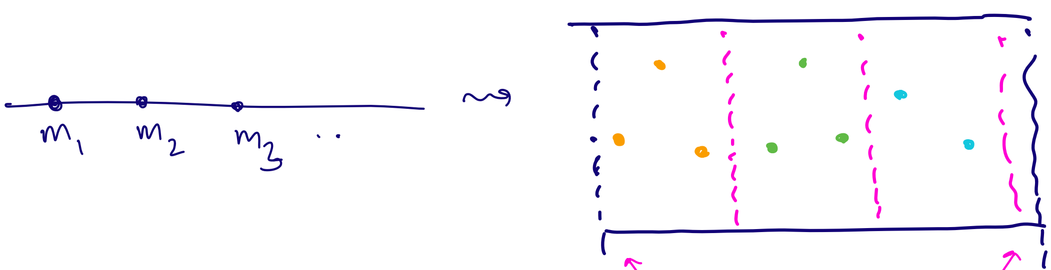
Also a path:



Can do this construction for any monoid M.

$$M = \coprod_{n \geq 0} M_n \quad , \quad M_n \times M_m \longrightarrow M_{m+n}$$

Visualising BM for $M = \coprod_{n \geq 0} \mathrm{UConf}_n \mathbb{R}^2$:



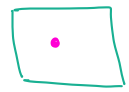
dotted lines indicate that points are allowed to disappear off these ends

III Group Completion

Suppose we have a monoid $M = \coprod_{n \geq 0} M_n$.

For a fixed $m \in M_1$, we get stabilisation maps

$$M_1 \xrightarrow{\times m} M_2 \xrightarrow{\times m} M_3 \xrightarrow{\times m} M_4 \to \cdots$$

(In our case, $M = \coprod \mathrm{UConf}_n \mathbb{R}^2$, $m =$ )

Let M_∞ = colimit of the above diagram

(in our case, M_∞ = space of any finite no. of configuration points in $\mathbb{R}^2$)

The stable homology is $\cong H_*(M_\infty)$

Thm [Group Completion; McDuff, Segal, '70s]

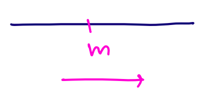
If $M = \coprod_{n \geq 0} M_n$ is a monoid, and is homotopy commutative,

then $H_*(\mathbb{Z} \times M_\infty) \cong H_*(\Omega BM)$

(Thus $H_*(M_\infty) \cong H_*(\Omega_0 BM)$)

Can prove this theorem using:

Thm: If M is a monoid s.t. $\pi_0(M)$ is a group, then $M \simeq \Omega BM$.

The map is $m \longmapsto$  loop where m travels from left to right end

The reason for taking $\mathbb{Z} \times M_\infty$ in the Group Completion Thm is to make π_0 into a group.

$$\begin{tikzcd}
M_0 \arrow[dr, "\times m"] & M_0 \arrow[dr] & M_0 \arrow[dr] & \cdots \\
M_1 \arrow[dr] & M_1 \arrow[dr] & M_1 \arrow[dr] & \cdots \\
M_2 \arrow[dr] & M_2 & M_2 & \cdots \\
& & &
\end{tikzcd} \simeq \mathbb{Z} \times M_\infty$$

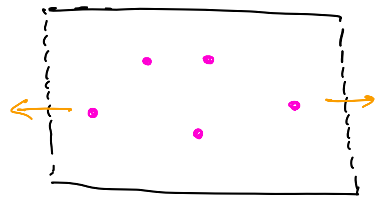
IV The Scanning Map

Thm: The stable H_* of $\mathrm{UConf}_n \mathbb{R}^2$ is $\cong H_*(\Omega^2_0 S^2)$

We shall prove this by applying Group Completion twice.

$M = \coprod_n \mathrm{UConf}_n \mathbb{R}^2$

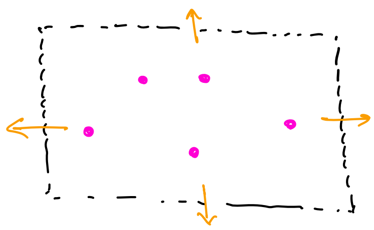
BM :



points can disappear off the left & right ends

$M' = BM$ itself has a monoidal structure, given by stacking vertically

BM' :



points can disappear both vertically and horizontally

Let $M'' = BM'$.

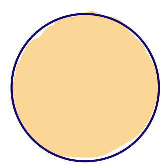
$$\text{Stable } H_* \text{ of } \mathrm{UConf}_n \mathbb{R}^2 \overset{\text{Gp Completion}}{=\!=} H_*(\Omega BM) = H_*(\Omega M')$$

$$\overset{\text{Gp Completion}}{=\!=} H_*(\Omega \Omega BM')$$

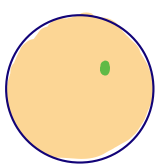
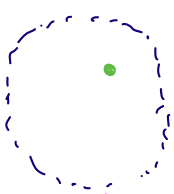
$$= H_*(\Omega^2 M'')$$

Prop: $M'' \simeq S^2$

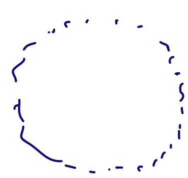
Pf: Think of $S^2 = D^2/\partial D^2$



$S^2 \longrightarrow M''$

 $\mapsto$ 

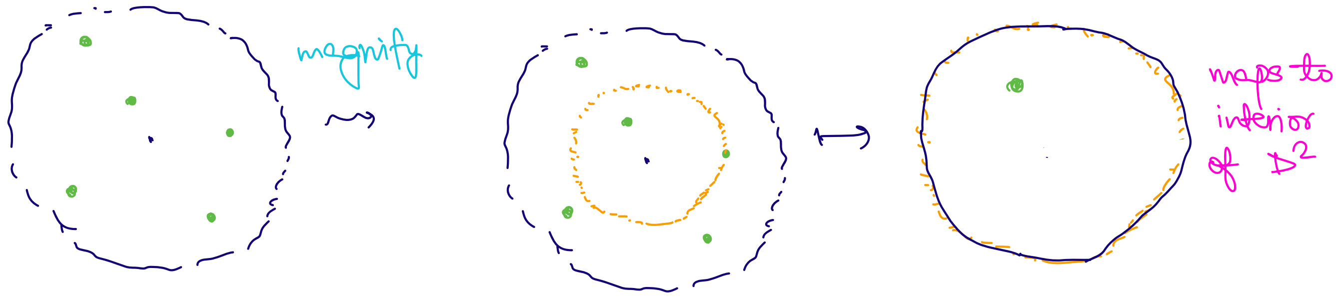
single-point configuration

$\partial D^2 \longmapsto$ 

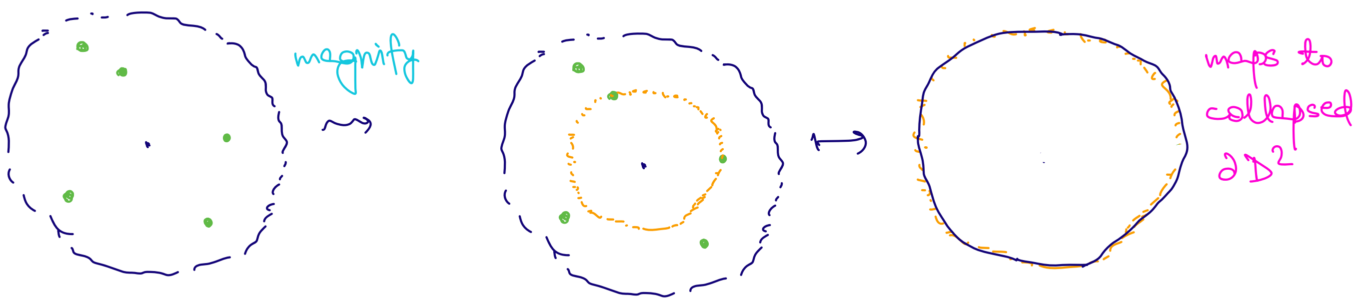
empty configuration

$M'' \longrightarrow S^2$

Case 1 : There is a unique closest point to the center

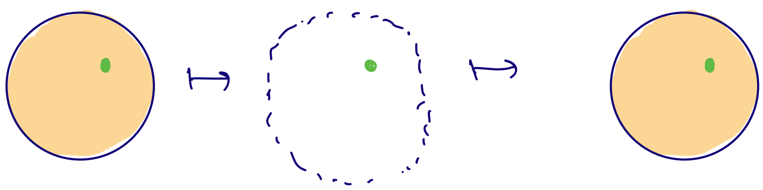


Case 2 : There are two or more closest points to the center

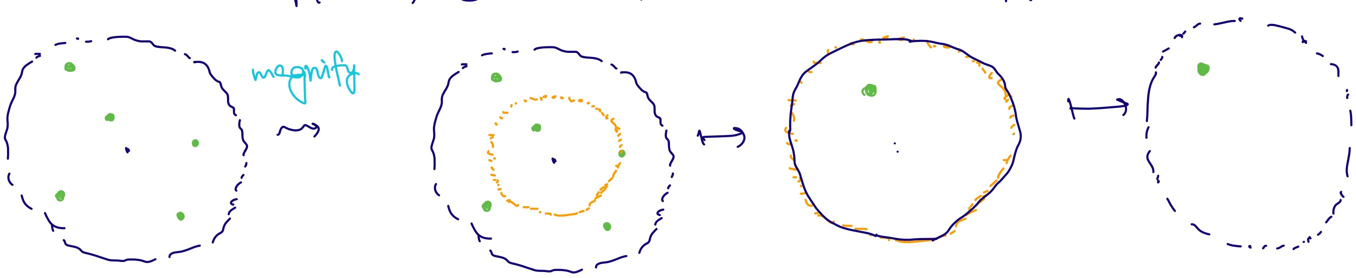


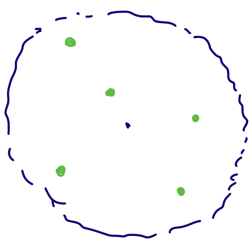
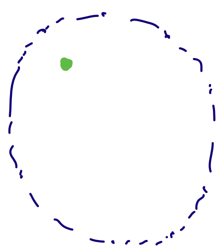
Note that :

- $S^2 \longrightarrow M'' \longrightarrow S^2$ is id_{S^2}



- $M'' \longrightarrow S^2 \longrightarrow M''$ is $\simeq id_{M''}$



 can be homotoped to  by radially projecting all the green points outward until only one is left.

Day 5 : The Little Disks Operad

I) Operad : Encodes ways of multiplying things

$$\mathcal{O} = \coprod_{n \geq 0} \mathcal{O}(n)$$

$\mathcal{O}(n)$: topological space
gives ways to multiply n things

$$\mathcal{O}(n) \times \mathcal{O}(i_1) \times \mathcal{O}(i_2) \times \dots \times \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + i_2 + \dots + i_n)$$

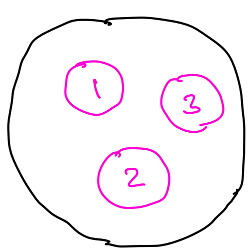
satisfying unit, associativity, equivariance axioms

$$\mathcal{O}(3) \otimes \mathcal{O}(2) \otimes \mathcal{O}(1) \otimes \mathcal{O}(4) \to \mathcal{O}(7)$$

The Little Disks Operad

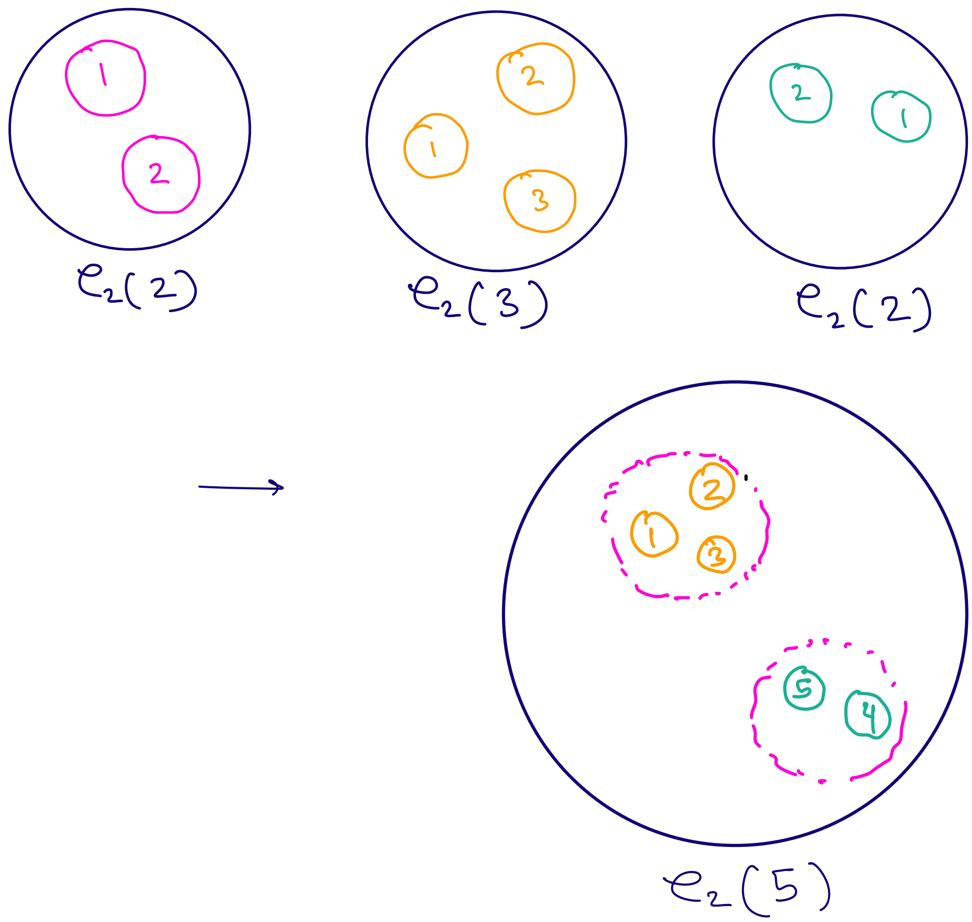
$$e_2 = \coprod e_2(n)$$

$e_2(n)$: embeddings of n disks in the unit disk



$\in e_2(3)$

$$e_2(n) \times e(i_1) \times \dots \times e(i_n) \to e(i_1+i_2+\dots+i_n)$$



II Algebra over an Operad

$\mathcal{O} = \coprod \mathcal{O}(n)$

A : topological space

$$\mathcal{O}(n) \times \underbrace{A \times A \times \dots \times A}_{n} \to A$$

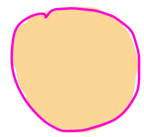
satisfying axioms . . .

Eg: $\mathcal{O} = e_2$ Little disks operads

$\Omega^2 X = ((S^2, *) \to (X, *))$

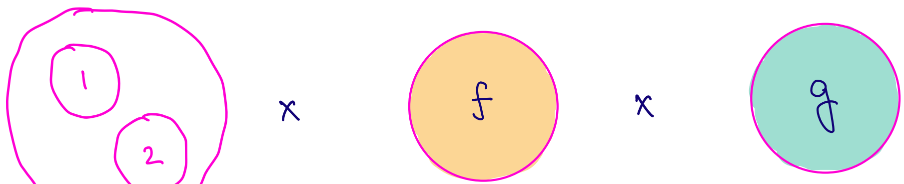
$$e_2(n) \times \underbrace{\Omega^2 X \times \dots \times \Omega^2 X}_{n} \to \Omega^2 X$$

$S^2 = D^2 / \partial D^2$



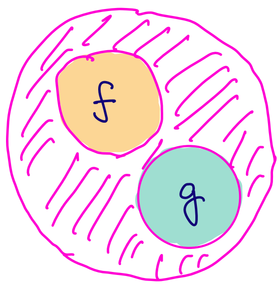
Think of $(S^2, *) \longrightarrow (X, *)$ as a map $D^2 \longrightarrow X$ which sends

$$\partial D^2 \longmapsto *$$



$e_2(2)$ $\quad$ $f : S^2 \to X$ $\quad$ $g : S^2 \to X$

$\longrightarrow$

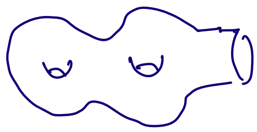


Pink shaded portion $\mapsto * \in X$

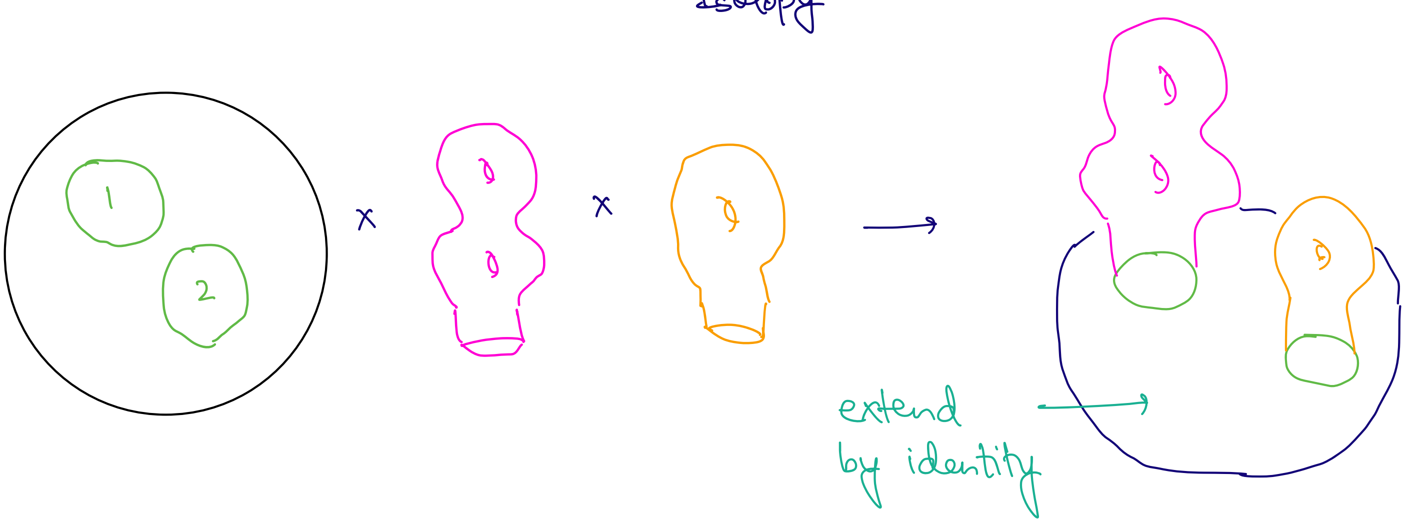
$h : S^2 \to X$

Eg: <u>Mapping Class Groups</u>

$\Sigma_{g,1}$:



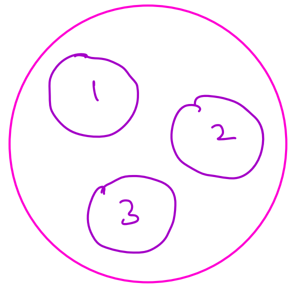
$$\mathrm{Mod}(\Sigma_{g,1}) = \frac{\mathrm{Diffeo}(\Sigma_{g,1}, \partial \Sigma_{g,1})}{\mathrm{Isotopy}}$$



III The Homology of the Little Disks Operad

$$e_2 = \coprod e_2(n)$$

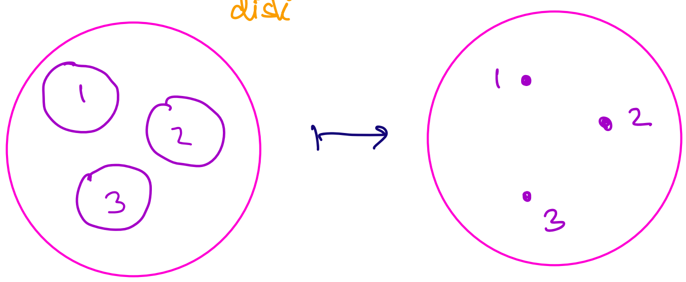
$$H_*(e_2(n))$$



$\in e_2(3)$

$$e_2(n) \longrightarrow \mathrm{Conf}_n(\mathrm{Int}\, D^2)$$

records center of each disk



This is a fibration with contractible fibers.

$$\Rightarrow \quad H_*(e_2(n)) \cong H_*(\mathrm{Conf}_n(\mathrm{Int}\, D^2)) \cong H_*(\mathrm{Conf}_n \mathbb{R}^2)$$

<u>Prop^n</u>: If $\mathcal{O}$ is an operad, then $H_*(\mathcal{O})$ is an operad

$$H_*(\mathcal{O}(n)) \otimes H_*(\mathcal{O}(i_1)) \otimes \cdots \otimes H_*(\mathcal{O}(i_n)) \xrightarrow{\text{Künneth}} H_*(\mathcal{O}(n) \times \mathcal{O}(i_1) \times \cdots \times \mathcal{O}(i_n)) \longrightarrow H_*(\mathcal{O}(i_1 + i_2 + \cdots + i_n))$$

If A is an algebra over $\mathcal{O}$, then $H_*(A)$ is an algebra over $H_*(\mathcal{O})$.

$$H_*(\mathcal{O}(n)) \otimes H_*(A) \otimes \cdots \otimes H_*(A) \xrightarrow{\text{Künneth}} H_*(\mathcal{O}(n) \times A \times \cdots \times A) \longrightarrow H_*(A)$$

So $H_*(A)$ has a lot of extra structure.
We can better understand $H_*(A)$ by understanding $H_*(\mathcal{O})$.

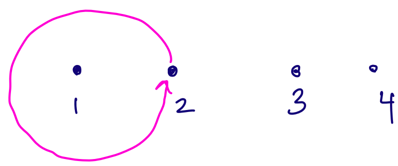
So it's useful that in the case of the little disks operad, we know its homology entirely.

Closing Remarks: $\mathbb{R}^2$ versus $\mathbb{R}^d$

Everything from this week goes through for $\mathbb{R}^d$

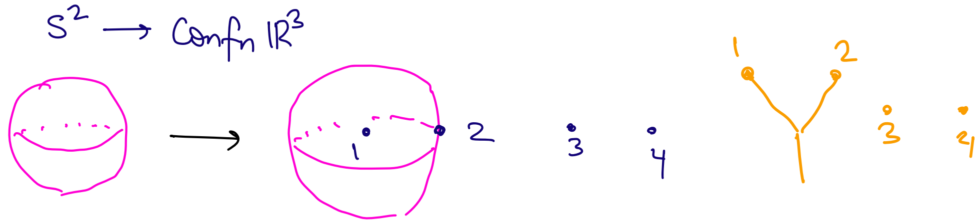
- $H_*(\mathrm{Conf}_n \mathbb{R}^d) \rightsquigarrow$ orbit systems
- Scanning $\rightsquigarrow$ stable H_*
- Little disks operad

$\underline{H_1(\mathrm{Conf}_n \mathbb{R}^3)}$ $\cong 0$



$\simeq$ constant loop

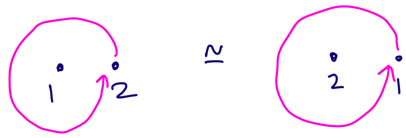
$\underline{H_2(\mathrm{Conf}_n \mathbb{R}^3)}$ $\quad S^2 \longrightarrow \mathrm{Conf}_n \mathbb{R}^3$



$\underline{H_4(\mathrm{Conf}_n \mathbb{R}^3)}$ $\quad S^2 \times S^2 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$

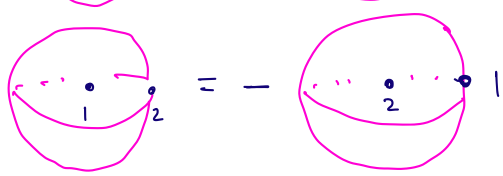
$\mathrm{Conf}_n \mathbb{R}^3$ has non-trivial H_* only in even degrees, generated by orbit systems
Similarly, $\mathrm{Conf}_n \mathbb{R}^2$ has non-trivial H_* in degrees $(d-1), 2(d-1), 3(d-1), \ldots$

In $\mathbb{R}^2$:



$x \mapsto -x$ in S^1
orientation-preserving

In $\mathbb{R}^3$:



$x \mapsto -x$ in S^2
orientation-reversing