Day 1

1. X - A Riemann surface (1-dim cmplx mfld)

If X is compact, then we know that $X \underset{\text{homeo}}{\cong} \Sigma_g$

<u>Uniformization Thm</u>: X - simply conn R. surface
Then X bihol. to $\mathbb{C}$, $\mathbb{D}$ or $\hat{\mathbb{C}}$

unit disk w/o bdry

$\hat{\mathbb{C}} = \mathbb{P}^1$

$\therefore$ If X is a R. surface, $\tilde{X}$ universal cover, then $\tilde{X} \in (\hat{\mathbb{C}}, \mathbb{P}^1, \mathbb{C})$

$$\begin{array}{c}\tilde{X}\\ \downarrow \\ X\end{array}$$

So every R. surface can be realised as a quotient of $\mathbb{C}$, $\mathbb{P}^1$ or $\mathbb{D}$

Consequence: <u>Fact</u>: $\mathbb{P}^1 \to X$ → Can show that X has to be $\mathbb{P}^1$ for this to happen

covering maps

$\mathbb{C} \to X$

Consider Deck transformations. They have to be a discrete family of bihol. maps.

Bihol. maps $\mathbb{C} \to \mathbb{C}$: $\{az+b \mid a, b \in \mathbb{C}\}$

Any discrete family of them looks like either $\langle a_1 z + b_1 \rangle$ or $\left\langle \begin{matrix} a_1 z + b_1 \\ a_2 z + b_2 \end{matrix} \right\rangle$, where $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ has $\neq 0$ det.

Quotienting $\mathbb{C}$ by the actions of either of these gives a <u>cylinder</u> or <u>torus</u>, respectively.

($\underset{\text{homeo}}{\cong} S^1 \times \mathbb{R}$)

So <u>Uniformization</u> $\Rightarrow$ If X is a R. surface, not $\mathbb{P}^1$, Cyl. or Torus or $\mathbb{C}$, then can be realised as a quotient of $\mathbb{D}$.

Defn: (Branched Cover) X, Y Riemann surfaces

$f : X \to Y$ s.t. $\exists\, B \subset Y$, $|B| < \infty$
s.t., for $A = f^{-1}(B)$

$f|_{X \setminus A} : X \setminus A \to Y \setminus B$ is a covering

If B is minimal, $B_f := B$ is called the set of branch pts of f.

$R_f := f^{-1}(B)$: ramification points of f

Examples: 1) Polynomials: $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$

2) If X, Y cpct, $f : X \to Y$ non-constant hol.
Then f is a branched cover of finite degree

Defn: $f : X \to Y$. X, Y compact top space
f is "proper" if $\forall\ \underline{K << Y}$,
$\underline{f^{-1}(K) << X}$ } (compact subset notation)

Eg: • $\mathbb{C} \to \mathbb{C} \setminus \{0\}$
$f(z) = e^z$ is an infinite sheeted branched cover

• Any entire fn: $\mathbb{C} \to \mathbb{C}$ is a branched cover

The Riemann - Hurwitz Formula

If X, Y are cpct R. surfaces

$f : X \to Y$ has degree N

r - # ramification pts

b - # branch pts

Then

$$\chi(X) - r = N(\chi(Y) - b)$$

Pf : 1. Triangulation technique to show $\chi(X) = N\chi(Y) - \sum_{p \in X}(\deg_f(p) - 1)$

2. Note that $\sum_{p \in X}(\deg_f(p) - 1) = Nb - r$, and thus the formula follows.

The Triangulation Technique :

Σ : Triangulation of Y — $\begin{cases} v & \text{vertices} \\ e & \text{edges} \\ f & \text{faces} \end{cases}$

Assume all branch points coincide with vertices

Σ' : Pullback* of Σ by $f : X \to Y$. Then, Σ' has : Nf faces

Ne edges

$Nv - \sum_{p \in X}(\deg_f(p) - 1)$ vertices

Now,

$$\chi(Y) = f - e + v$$

$$\chi(X) = Nf - Ne + Nv - \sum_{p \in X}(\deg_f(p) - 1)$$

$$\Rightarrow \boxed{\chi(X) = N\chi(Y) - \sum_{p \in X}(\deg_f(p) - 1)}$$

* Have to be careful about "lifting" the triangulation Σ — I think the fact that the branch pts are at vertices, and the fact that f is a cover and hence a local homeomorphism, are important here to ensure that $f^{-1}(\Sigma)$ is indeed a triangulation! For instance, it ensures that edges pull back to edges, and no "breaking" or "branching" of the edge occurs. Or maybe this would be ensured just by f being a hol. and thus differential map? So that $f^{-1}(\text{edge})$ is also a mfld, and hence an edge? ← Mfld would have dim ≥ 1 though. Don't know for sure its $= 1$

Basically we need to count the # of pre-images of every pt.

Since this is a degree N map, we have N pre-images for non-branch pts, and $N - $ (# of repetitions) for branch pts. #rep's

$= \sum(\text{multiplicity} - 1)$

$= \sum_p(\deg f - 1)$

Exercises :

1) X, Y tori

$f : X \to Y$

$r = ?$

$b = ?$

$\chi(X) - r = N(\chi(Y) - b)$

$\chi(X) = \chi(Y) = 0$ unless we have something like local diffeo

$\Rightarrow r = Nb$

But (assuming the map is surjective) this is not possible. Any branch pt has N pre-images with multiplicity, and it being a branch pt implies atleast one of those multiplicities is >1 (for eg in the deg 2 map $z \mapsto z^2$, 0 has 1 pre-image, 2 w/ multiplicity) So the contribution of each branch pt to the set of ramification pts is $< N$. So, if $b > 0$, then $r < Nb$, a contradiction

Thus $r = b = 0$

2. $X = \mathbb{C}/\Lambda_1$, $Y = \mathbb{C}/\Lambda_2$ $(\Lambda_1 = \mathbb{Z} \oplus \tau_1 \mathbb{Z},\ \Lambda_2 = \mathbb{Z} \oplus \tau_2 \mathbb{Z})$
$f : X \to Y$. X, Y both have $\mathbb{C}$ as univ. cover. Can precompose f with $\mathbb{C} \to X$ to get $\mathbb{C} \to Y$, and then lift that to $g : \mathbb{C} \to \mathbb{C}$. Qn: What does g look like?

$$\begin{tikzcd} g: \mathbb{C} \arrow[r, dotted] \arrow[d] & \mathbb{C} \arrow[d] \\ f: X \arrow[r] & Y \end{tikzcd}$$

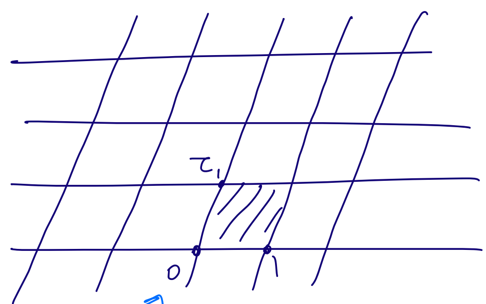
Need:
$$g(z+1) = g(z) + \lambda_2(z)$$
$$g(z+\tau_1) = g(z) + \omega_2(z)$$

$\lambda_2(z), \omega_2(z) \in \Lambda_2$

But note λ_2, ω_2 cts fns, taking values in discrete lattice Λ_2. So constant.

$$g(z+1) = g(z) + \lambda_2$$
$$g(z+\tau_1) = g(z) + \omega_2$$

$$\Rightarrow \quad g'(z+1) = g'(z)$$
$$g'(z+\tau_1) = g'(z)$$



This implies that the values of g' only come from this (compact) shaded region.
Since this region is cpct, these values are bounded

So, g' is an entire fn, with bdded range

$$\Rightarrow g' = C \quad (\text{constant})$$

$$\Rightarrow g = Cz + D.$$

$$\left.\begin{aligned} g(z+1) &= g(z) + \lambda_2 \\ g(z+\tau_1) &= g(z) + \omega_2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} Cz + C + D &= Cz + D + \lambda_2 \\ Cz + C\tau_1 + D &= Cz + D + \omega_2 \end{aligned}$$

$$\Downarrow$$

$C = \lambda_2 \in \Lambda_2$
$C\tau_1 = \omega_2 \quad \Lambda_2$

So, given $\Lambda_1 = \mathbb{C} \oplus \mathbb{C}\tau_1$, $\Lambda_2 = \mathbb{C} \oplus \mathbb{C}\tau_2$,

$$g \in \left\{ Cz + D \;\middle|\; \begin{array}{l} C \in \Lambda_2 \text{ and} \\ Dz \in \Lambda_2 \end{array} \right\}$$

Questions

→ Well-definedness of lifting up a triangulation

- Uniformization $\Rightarrow$ Little Picard

Little Picard: A non-constant holomorphic fn $f : \mathbb{C} \to \hat{\mathbb{C}}$ can miss atmost one point (i.e. there is atmost one point in $\hat{\mathbb{C}} \setminus \text{image} f$)

Proof: Suppose $\exists\, f : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{a, b\}$.

Uniformization $\Rightarrow$ univ. cover of $\hat{\mathbb{C}} \setminus \{a, b\}$ has to be $\mathbb{D}$

($\mathbb{P}^1$ can only cover $\mathbb{P}^1$; $\mathbb{C}$ can only cover tori and cylinders)

Lift f to $\tilde{f} : \mathbb{C} \to \mathbb{D}$

$$\begin{tikzcd} & \mathbb{D} \arrow[d] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \hat{\mathbb{C}} \setminus \{a, b\} \end{tikzcd}$$

$\tilde{f}$ is a bounded entire function. $\therefore$ Constant.

Contradiction.

Day 2

Monodromy Permutation Representations

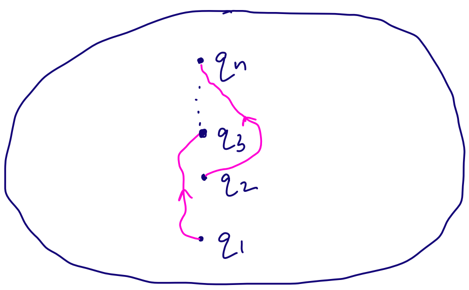
$f : X \xrightarrow{n} Y$ branched cover ; $B \subset Y$ branch points , $R \subset X$ ramification points

Fix $q \in Y \setminus B$. Let $\{q_1, \dots, q_n\} = f^{-1}(\{q\})$.

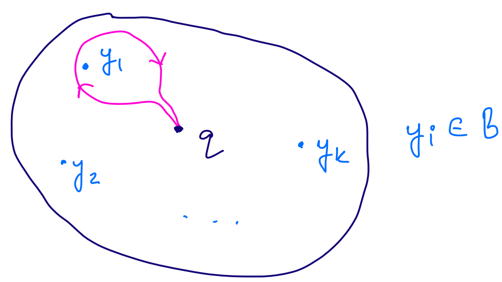
Take any loop $\gamma \in \pi_1(Y \setminus B, q)$. Lift γ starting at q_i , let the endpoint of this lift be $\sigma(i)$.

Thus lifting γ defines a permutation of $\{q_1, \dots, q_n\}$ — $i \longmapsto \sigma(i)$.

Thus, $f : X \to Y$ gives rise to a homomorphism $\rho : \pi_1(Y \setminus B, q) \to S_n$



X



Y

$\sigma(1) = 3$

$\sigma(2) = n$

Let $B = \{y_1, \dots, y_k\}$. For $1 \le i \le k$, let γ_i be a loop centred at q wrapping once around y_i .

We are particularly interested in the permutations $\rho(\gamma_i)$ defined by lifts of the γ_i .

(Thus this gives a way of studying how f "wraps loops around" the singularity y_i in $Y \setminus B$)

Eg:

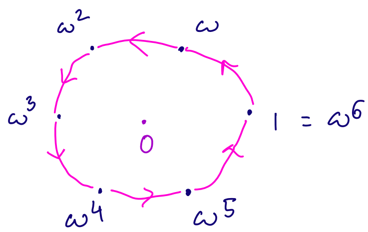
1) $f : \hat{\mathbb{C}} \xrightarrow{n} \hat{\mathbb{C}}$

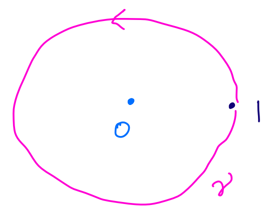
$z \longmapsto z^n$

$R = \{0, \infty\} = B$

Let $q = 1$.

Then $q_1 = e^{2\pi i/n} = \omega$, $q_2 = \omega^2$, $\ldots$, $q_n = 1$





$\varphi(\gamma) = (1\ 2\ 3\ \ldots\ n) \in S_n$

2) $f : \hat{\mathbb{C}} \xrightarrow{3} \hat{\mathbb{C}}$

$$z \longmapsto \frac{z^3}{(1-z)^2}$$

$$f'(z) = \frac{3z^2(1-z)^2 + 2z^3(1-z)}{(1-z)^4}$$

$$= \frac{3z^2(1-z) + 2z^3}{(1-z)^3} = \frac{3z^2 - 3z^3 + 2z^3}{(1-z)^3} = \frac{3z^2 - z^3}{(1-z)^3}$$

$$= \frac{z^2(3-z)}{(1-z)^3}$$

Critical points : $0, \ 3$

Critical values : $0, \ 27/4 \quad (= B)$

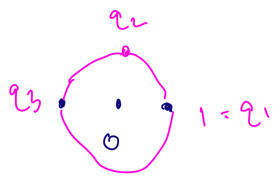
$f^{-1}(0) = 0$

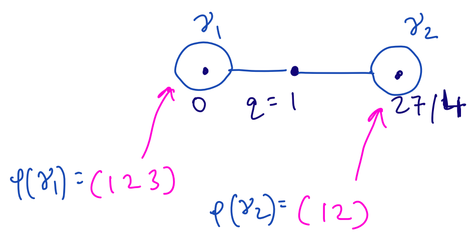
$f^{-1}(27/4)$

Local deg at 3 = (multiplicity of 3 in $f'(q)$) + 1

= 2. So one more pre-image

So $f^{-1}(27/4) = \{3, x\}$





<u>Observations</u> : 1) $\varphi : \pi_1(Y \setminus B, q) \to S_n$ defines a transitive action.

(Given $1 \le i, j \le n$, take any path γ from q_i to q_j. Then $f(\gamma)$ gives a loop centred at q, whose lift is γ. Thus $\varphi(f(\gamma))$ sends $i \mapsto j$)

2) Eg 2 shows that φ may not be <u>free</u>. ($\varphi(\gamma_2) = (1\,2) \neq id$, but $\varphi(\gamma_2)$ still fixes q_3)

(i.e. $\forall x$, $\{g : gx = x\} = \{id\}$)

But in Eg 1 it is free.

3) The action may or may not be <u>faithful</u> (although both examples here are faithful)

(i.e. $\{g \mid gx = x \ \forall x\} = \{id\}$)

Theorem: Fix Y, $B \subset Y$, $|B| < \infty$, $n \in \mathbb{N}$, $q \in Y \setminus B$.

Then we have a bijective correspondence:

$$\left\{ \begin{array}{l} \text{branched covers } f: X \xrightarrow{n} Y, X \text{ compact} \\ \text{branch pts } B_f \subset B \end{array} \right\} \Big/ \text{isomorphisms} \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d, shift left] & Y \\ X' \arrow[u, shift left] \arrow[ur, "f'"'] & \end{tikzcd} \right)$$

$$\downarrow \quad \uparrow$$

$$\{ \rho : \pi_1(Y \setminus B, q) \to S_n, \ \rho \text{ transitive} \} / \text{conjugation}$$

Proof: We already know how to get ρ from a given $f: X \to Y$.

So we need to construct a way to go in the the reverse direction.

From covering space theory: $\{ p: W \to Y \setminus B \text{ covering map} \}$

$$\uparrow \downarrow$$

$$\{ \text{Subgroups } H \subset G = \pi_1(Y \setminus B, q) \}$$

The subgroup H is precisely the group $p_*(\pi_1(W)) \subset \pi_1(Y \setminus B)$ (i.e. the loops in $Y \setminus B$ that get lifted to loops in W).

Another way saying this is H consists of the loops that define the trivial permutation on $\{1, .., n\}$ under ρ.

Let $H = \ker \rho \subset \pi_1(Y \setminus B, q)$.

Let $f: W \to Y \setminus B$ be the covering space associated to H.

Transitivity of $\rho \Rightarrow n$-sheeted cover

Can use the covering map f to lift the complex structure on $Y \setminus B$ to give a complex structure to W, thereby making it a Riemann surface.

In fact,

Fact: If $f: S \to Y$, where Y is a Riemann surface, S - Hausdorff, connected
f is a branched cover
Then $\exists$ complex structure on S which makes f hol.

We need to extend this cover $f: W \to Y \setminus B$ to a branched cover $\tilde{f}: \tilde{W} \to Y$.

For $y \in B$, pick a small disk D around y.

Then $f^{-1}(D \setminus \{y\}) = \coprod W_i$, where each $f|_{W_i}: W_i \to D \setminus \{y\}$ is a (finite sheeted) cover.

$D \setminus \{y\}$ is bihol. to $D \setminus \{0\}$ (just scale by a suitable factor)

In fact,

Fact: Two annuli of radii (R, r), (R', r') respectively are bihol $\Leftrightarrow$ $R/r = R'/r'$

$D \setminus \{0\} \cong D \setminus \{y\}$ is the special case of this with $r = r' = 0$

<u>Fact</u> : Any finite-sheeted cover of $\mathbb{D}\setminus\{0\}$ has to be by a $\mathbb{D}\setminus\{0\}$ (the covering map being the usual $z \mapsto z^k$)

This can be proved using the Uniformization Thm.
$\mathbb{D}\setminus\{0\}$ is homeo (and bihol) to a cylinder, and so its covers are $\mathbb{C}$ (infinite-sheeted) or $\mathbb{D}\setminus\{0\}$ itself.

So, we have :
$$\begin{array}{ccc} \mathbb{D}\setminus\{0\} & \rightleftarrows & W_i \\ \text{deg } k \downarrow & & \downarrow k \\ \mathbb{D}\setminus\{0\} & \leftrightarrows & \mathbb{D}\setminus\{y\} \end{array}$$

Lifting the $\mathbb{D}\setminus\{0\} \to \mathbb{D}\setminus\{y\}$ bihol. gives rise to a bihol. $\mathbb{D}\setminus\{0\} \to W_i$.

Use this to "add a point" to W_i by extending the map to $\mathbb{D} \to W_i \cup \{b_i\}$.

Extend the map $f : W \to Y\setminus B$ to $\tilde{f} : W \cup \{b_i\} \to Y$ by letting $\tilde{f}(b_i) = y$.

$X := W \cup \{\text{new disks}\}$ is the desired compact branched cover of Y

<u>Eg.</u> $Y = \hat{\mathbb{C}}$

$B = \{0, \infty\}$

$g = 1$

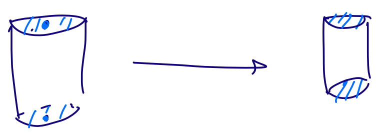
$Y \setminus B \cong$ cylinder

$\rho : \pi_1(Y \setminus B) = \mathbb{Z} \longrightarrow S_n$

$1 \longmapsto (1\ 2\ \cdots\ n)$

$\ker \rho = n\mathbb{Z}$

Corresponding cover: cylinder (wraps around n times)

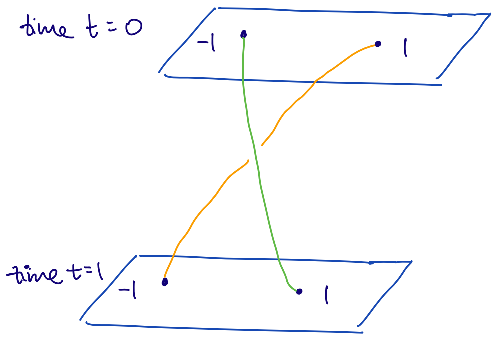


Extending the map from $\hat{\mathbb{C}} \setminus \{0,\infty\} \to \hat{\mathbb{C}} \setminus \{0,\infty\}$ to $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ using the above construction gives back the branched cover $z \mapsto z^n$.

<u>Aside</u> : Monodromy Braid Representations

Consider the example $z \mapsto z^2$.

The permutation determined by the lift(s) of the loop γ is $(1\ 2)$.
If in addition, we also traced the <u>motion</u> of the lifts γ_1 and γ_2 w.r.t. each other:



We get a braid. This is called the monodromy braid representation of f.

Similar to how braids give more information by encoding the "tangling up" of strands rather than just the permutation they define, monodromy braid representations can encode more information than monodromy permutation representations.

Applications :

- Monodromy Permutation Reps : Studying solvability of polynomials ;

- Monodromy Braid Reps : Curt McMullen's "Braiding of the Attractor"

Day 3

Hyperelliptic Curve : A compact Riemann surface X s.t.
$\exists$ holomorphic map (and hence branched cover)
$f : X \xrightarrow{2} \mathbb{P}^1$ of degree 2.

Let g: genus of X, b: no. of branch pts, r: no. of ramification points.

We always have $b \leq r \leq (n-1)b$. Since $n = 2$ here, this forces $b = r$.

Riemann - Hurwitz formula $\Rightarrow$

$$\chi(X) - b = 2(\chi(\mathbb{P}^1) - r)$$

$$\Rightarrow 2 - 2g - b = 4 - 2r$$

$$\Rightarrow 2r - b = 2 + 2g$$

$$\Rightarrow \boxed{b = 2 + 2g = r}$$

Thus $\pi : X \xrightarrow{2} \mathbb{P}^1$ determines a tuple of $2g+2$ pts on $\mathbb{P}^1$.

Fact : The opposite is also true.

We will sketch a rough proof of this.

Let $a_1, a_2, \dots, a_{2g+2} \in \mathbb{P}^1$

$h(x) := (x - a_1)(x - a_2) \dots (x - a_{2g+2})$

Let $X := \{(x, y) \mid y^2 = h(x)\}$

X is a Riemann surface.

However X is not necessarily compact — ~~yet~~.

Let $k(z) := z^{2g+2} h(1/z)$

Zeroes of k : $1/a_i$, $1 \le i \le 2g+2$

Let $Y := \{(z, w) \mid w^2 = k(z)\}$

Define $\varphi : X \setminus \{(x,y) \mid x = 0\} \to Y$ as follows:

$$U := X \setminus \{(x,y) \mid x = 0\} \longrightarrow Y$$
$$(x, y) \longmapsto (1/x \, , \, w)$$

Need : $w^2 = k(1/x) = \frac{1}{x^{2g+2}} h(x) = \frac{y^2}{x^{2g+2}}$

$\Rightarrow w = \pm \frac{y}{x^{g+1}}$.

Choose $+\frac{y}{x^{g+1}}$

So,

$$\varphi : \underbrace{X \setminus \{(x,y) \mid x = 0\}}_{= U} \longrightarrow Y$$
$$(x, y) \longmapsto \left(1/x \, , \, \frac{y}{x^{g+1}}\right)$$

$\operatorname{Im} \varphi =: V$

Let $\boxed{Z = U \sqcup V / \{(x,y) \sim \varphi(x,y)\}}$

Turns out — Z is compact
Z has genus g
Z is the hyperelliptic surface we're looking for.

(Define $\pi : Z \to \mathbb{P}^1$ by $(x,y) \mapsto x$)

Hyperelliptic Involution

$\varphi : Z \to Z$
$(x,y) \mapsto (x,-y)$

Then
- φ has order 2
- φ fixes $2g+2$ pts (i.e. the points $(a_i, 0)$)

φ : Hyperelliptic involution

Defn : Z : cpct Riemann surface.
$\varphi : Z \to Z$ is a hyperelliptic involution if
- φ has order 2, and
- φ fixes $2g+2$ points.

Fact : Z is a hyperelliptic surface

$\Updownarrow$

Z has a hyperelliptic involution

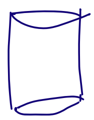
- The $\Uparrow$ can be proved by quotienting Z by the action of the involution φ to get the map $\pi : Z \to \mathbb{P}^1$
- The $\Downarrow$ direction can be proved by using the above construction and the further following fact :

Fact : Two tuples $(a_1, a_2, .., a_{2g+2})$ and $(a_1', a_2', .., a_{2g+2}')$ in $\mathbb{P}^1$ give rise to isomorphic Z, Z'
$\iff$ The tuples are related by a Möbius transformation.

Moduli Spaces – Examples

$\mathrm{Mod}(S^2) = \{*\}$ $\quad (\mathbb{P}^1)$

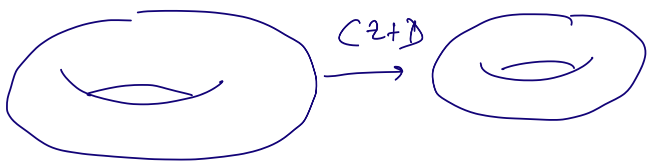
$\mathrm{Mod}(\mathbb{D}) = \{x, y\}$ $\quad (\mathbb{D}, \mathbb{C})$

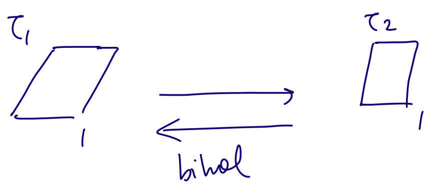


$\mathbb{C}, \quad \mathbb{D}, \quad \mathbb{P}^1$

Ex: What is $\mathrm{Mod}(\mathbb{D} \setminus \{x\})$ (Ans: $\{\mathbb{C}^\times, \mathbb{D}^\times, (0, \infty)\}$

The $\frac{R}{r}$ ratio for an annulus)

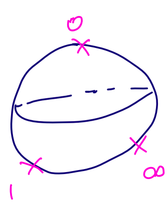




$\Leftrightarrow \tau_1 = \tau_2$

So $\mathrm{Mod}(\text{Torus})$ parameterized by $\mathbb{H}$

Sphere w/ 3 punctures



X

$\mathbb{D}$

$\downarrow$

X

$\mathrm{Mod}(S^2 \setminus \{0, 1, \infty\}) = \{*\}$

$\mathrm{Mod}(S^2 \setminus \{a, b, c, d\}) = \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$

$\mathrm{Mod}(S^2 \setminus n \text{ pts}) \subset \mathbb{P}^{n-3}$