Grassmannian Cohomology and Symmetric Polynomials

(I) Motivation & Definitions

A problem in enumerative geometry :

Fix four lines in $\mathbb{CP}^3$ in general position. How many lines intersect all four of them?

Ans: 2 (!)

We can answer this by understanding $H^*(Gr(2,4))$

Defn : Let $Gr(k,n)$ denote the space of k-dimensional linear subspaces of $\mathbb{C}^n$.

Eg: $\mathbb{CP}^n = Gr(1, n+1)$

Fact :
- (complex) dim $Gr(k,n) = k(n-k)$
  (real) dim $Gr(k,n) = 2k(n-k)$
- $Gr(k,n)$ is compact and orientable

(II) Topological Preliminaries

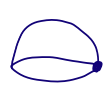
1. Computing (co)homology using cell structure

X : CW-complex

C_n : free abelian gp. generated by n-cells

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \to C_0 \to 0$$

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$



In particular, if all cells are even dim, $\partial_n = 0$. So $H_{2n}(X) \cong C_{2n}$.

2. Poincare duality & Intersection

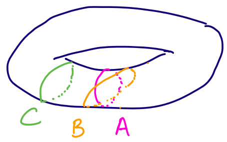
Thm : If X is a compact oriented manifold of dim n, then

$$H_i(X) \cong H^{n-i}(X)$$
$$[A] \mapsto [A]^*$$

Moreover, if A, B intersect transversely, then

$$[A]^* \smile [B]^* = [A \cap B]^*$$

Eg:



A B C represent the same homology class

A B intersect transversely

A C do not

$\to$ Will use this idea of intersection $\rightleftharpoons$ products throughout the talk.

III Cell Structure on $Gr(k,n)$

Fix a complete flag F :

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$$

where $F_i = \langle e_1, e_2, \ldots, e_i \rangle$ $\quad$ $\langle e_i \rangle$ some basis of $\mathbb{C}^n$

Have cells indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$
$(n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$

$$\sigma_\lambda(F) = \{ X \in Gr(k,n) \mid \dim(X \cap F_{n-k-\lambda_i+i}) = i, \ \dim(X \cap F_{n-k-\lambda_i+i-1}) = i-1 \quad \forall\ 1 \leq i \leq k \}$$

Eg: $Gr(2,5)$, $\underset{k}{2}$, $\underset{n}{5}$

$\lambda = (2,1)$ $\qquad$ $F_{5-2-2+1} = F_2$

$\qquad\qquad\qquad\quad$ $F_{5-2-1+2} = F_4$

$$\sigma_\lambda(F) = \{ X \mid \dim(F_2) = 1 \quad \dim(F_1) = 0$$
$$\dim(F_4) = 2 \quad \dim(F_3) = 1 \}$$

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} * & * & \cdots & 1 & 0 & \cdots & & 0 \\ * & * & \cdots & 0 & * & \cdots & 1 & 0 \cdots 0 \\ & & & & & & & 1 \end{bmatrix}$$

$\uparrow$ row reduced echelon form!

$$\overline{\sigma_\lambda(F)} = \{ X \mid \dim(X \cap F_{n-k-\lambda_i+i}) \geq i \}$$

$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}$$

- (complex) dim $\sigma_\lambda(F) = k(n-k) - |\lambda|$ $\qquad (|\lambda| = \lambda_1 + \lambda_2 + \cdots)$

  (real) dim $\sigma_\lambda(F) = 2k(n-k) - 2|\lambda|$

$H^{2m}(Gr(k,n)) \cong$ Free abelian gp generated by $[\sigma_\lambda(F)]$
s.t. $\ell(\lambda) \leq k$, $|\lambda| = \ell$

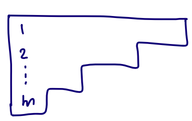
In fact this shows that as an abelian groups,

$$H^*(Gr(k,n)) \cong \bigoplus_{\substack{\lambda_1 \leq n-k \\ \ell(\lambda) \leq k}} \sigma_\lambda$$

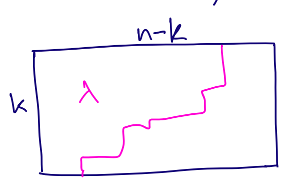
IV Multiplicative Structure on $H^*(Gr(k,n))$

Λ : ring of symmetric polynomials in $x_1, x_2, \ldots, x_m, \cdots$

Schur polynomials s_λ form a basis for Λ
correspond to SSYT



Theorem: $H^*(Gr(k,n)) \cong \Lambda / \langle s_\lambda \mid \lambda_1 > n-k \text{ or } \ell(\lambda) > k \rangle$



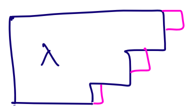
$$[\sigma_\lambda(F)] \mapsto s_\lambda$$

(Thm (Flag Invariance)): $[\sigma_\lambda(F)] = [\sigma_\lambda(F')]$

Outline of Proof : • $\sigma_\lambda \mapsto S_\lambda$ gives additive homomorphism, but need to check consistency with multiplication

• Pieri's Rule :

$$S_\lambda \cdot S_{(m)} = \sum_{\substack{|\lambda'| = |\lambda| + m \\ \lambda \uparrow \lambda'}} S_{\lambda'}$$



Strategy: Prove Pieri's rule for the σ_λ

$$\sigma_\lambda \cdot \sigma_{(m)} = \sum_{\substack{|\lambda'| = |\lambda| + m \\ \lambda \uparrow \lambda'}} \sigma_{\lambda'}$$

Proving Pieri's rule for σ_λ :

• Reverse flags : Given F, its reverse flag $\tilde{F}$ is given by
$\tilde{F}_i = \langle e_n, e_{n-1}, \dots, e_{n-i+1} \rangle$

Eg : $Gr(2,5)$
$\mu = (2,1)$

$$\sigma_\mu(\tilde{F}) = \{ X \mid \dim(X \cap \langle e_4, e_5 \rangle) = 1 \quad \dim(X \cap \langle e_5 \rangle) = 0$$
$$\dim(X \cap \langle e_2, e_3, e_4, e_5 \rangle) = 2 \quad \dim(X \cap \langle e_3, e_4, e_5 \rangle) = 1 \}$$

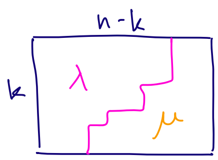
$$\begin{bmatrix} 0 & 0 & 0 & 1 & * \\ 0 & 1 & * & 0 & * \end{bmatrix}$$

(Contrast with $\sigma_{(2,1)}(F)$ $\begin{bmatrix} * & 1 & 0 & 0 & 0 \\ * & 0 & * & 1 & 0 \end{bmatrix}$

Not a coincidence! $(2,1) + (1,2) = (3,3) = (5-2, 5-2)$)

- **Duality Theorem:**

$$\sigma_\lambda(F) \cap \sigma_\mu(\tilde{F}) = \begin{cases} 1 \cdot \sigma_{(n-k\ n-k\ \dots\ n-k)} & \text{if } \lambda_i + \mu_{k-i+1} = n-k \ \forall\, i \\ 0 & \text{if } \lambda_i + \mu_{k-i+1} > n-k \text{ for any } i \end{cases}$$



$$[\sigma_\lambda] \smile [\sigma_\mu] = \begin{cases} 1 & \text{if } \lambda_i + \mu_{k-i+1} = n-k \\ 0 & \text{if } \lambda_i + \mu_{k-i+1} > n-k \text{ for any } i \end{cases}$$

"$\mu = \tilde{\lambda}$, dual to λ"

We wanted:

$$\sigma_\lambda \cdot \sigma_{(m)} = \sum_{\dots\dots} \sigma_{\lambda'}$$

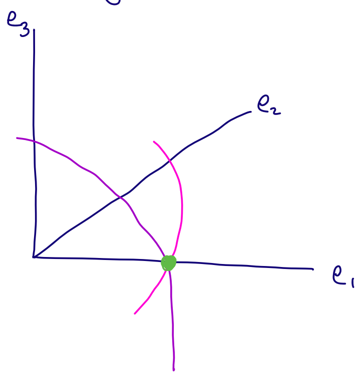
Multiply by $\sigma_{\tilde{\lambda}'}$:

$$\sigma_{\tilde{\lambda}'} \cdot \sigma_\lambda \cdot \sigma_{(m)} = \sigma_{\tilde{\lambda}'} \cdot \sigma_{\lambda'} = 1$$

Can prove this by studying intersections of cells

V Back to our original problem...

Lines intersecting in $\mathbb{CP}^3$ $\rightleftharpoons$ Planes intersecting in $\mathbb{C}^4$



e_1e_2 plane $\cap$ e_1e_3 plane

$= e_1$ - axis

Given a plane X in $\mathbb{C}^4$, can pick a basis $\overbrace{e_1, e_2, e_3, e_4}^{\text{Flag } F}$ s.t. $X = \langle e_1, e_2 \rangle$.

Let $\ell(X)$ be corresponding line in $\mathbb{CP}^3$.
Lines intersecting with $\ell(X)$ correspond to planes $Y \subset \mathbb{C}^4$ so that $\dim(Y \cap \langle e_1, e_2 \rangle) = 1$. Thus $\boxed{Y \in \sigma_{(1,0)}(F)}$.

$$\sigma_{(1,0)}(F^1) \cap \sigma_{(1,0)}(F^2) \cap \sigma_{(1,0)}(F^3) \cap \sigma_{(1,0)}(F^4)$$

Look at product of cohomology classes instead:

$$\sigma_{(1,0)} \cdot \sigma_{(1,0)} \cdot \sigma_{(1,0)} \cdot \sigma_{(1,0)}$$

$$= \underline{2} \cdot \sigma_{(2,2)}$$

Ans: 2 !