I Last Time:

① Defn: Gp Cohomology (Topologically)

There is an algebraic version, equivalent

Eg: $(\mathbb{Z}, S^1)$, $(\mathbb{Z}^2, S^1\times S^1)$, $(\mathbb{Z}_2, \mathbb{R}P^\infty)$

To find $K(G,1)$: wanted contractible $\tilde{X}$ with $G \curvearrowright \tilde{X}$ cov. space action

② Cohomological dimension $cd(G)$

Eg: $cd(\mathbb{Z}) = 1$, $cd(\mathbb{Z}^2) = 2$, $cd(\mathbb{Z}_2) = \infty$

→ can bound cd by bounding dim of $K(G,1)$

→ $cd(\mathbb{Z}_m) = \infty$

→ $H \subset G \Rightarrow cd(H) \le cd(G)$

Cor: G has torsion $\Rightarrow cd(G) = \infty$

③ $SL_n\mathbb{Z} \curvearrowright SL_n\mathbb{R}/SO(n) \cong \mathrm{Sym}_n/\{\text{scalar}\}$

dim $\binom{n+1}{2} - 1$, contractible

But can further shrink down to dim $\binom{n}{2}$

For $n = 2$,

$SL_2\mathbb{Z} \curvearrowright SL_2\mathbb{R}/SO(2) \cong \mathbb{H}^2$

Can cut this down to a 1-dim graph.

Problems:
- Not free action (eg. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ fixes i)
- $SL_n\mathbb{Z}$ has torsion (eg. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has order 4)

What helps:
- Finite stabilizers
- Torsion-free subgps of finite index
- $vcd(SL_n\mathbb{Z}) = \binom{n}{2}$ ...

Today: How this helps, and vcd.

II Virtual Cohomological Dimension

If $H \subset SL_n\mathbb{Z}$ torsion-free, then $H \curvearrowright SL_n\mathbb{R}/SO(n)$ freely ($H \cap \mathrm{Stab}(\sigma) = \{1\}$ $\forall \sigma$, since $\mathrm{Stab}(\sigma)$ is finite and thus only has torsion elements)

Finite index is what helps connect this back to $SL_n\mathbb{Z}$ ...

Serre's Thm: If G is torsion-free, $H \subset G$ s.t. $[G:H] < \infty$.
Then $cd(H) = cd(G)$.

Two important assumptions here: $\boxed{G \text{ torsion-free, and } [G:H] < \infty}$

Non-examples:

- $\{1\} \subset \mathbb{Z}_m$ — Finite index, but not torsion-free

  $\{1\}$: cd 0, $\mathbb{Z}_m$: cd ∞

- $\mathbb{Z} \subset \mathbb{Z}^2$, $n \mapsto (n, 0)$ — Torsion-free, but infinite index

  $\mathbb{Z}$: cd 1, $\mathbb{Z}^2$: cd 2

Sketch of Proof: Break into cases: ① $cd(G) < \infty$: Can prove $cd(H) = cd(G)$ algebraically (uses finite index assumption, and indirectly torsion-free 'coz $cd(G) < \infty$)

② $cd(G) = \infty$: If $cd(H) = \infty$, then done. Otherwise:

Lemma: G torsion-free, $H \subset G$ s.t. $[G:H] = n < \infty$.
Then $cd(H) < \infty \Rightarrow cd(G) < \infty$.

Pf Idea: ① Take a finite-dimensional CW-complex X' s.t. X' contractible, $H \curvearrowright X'$ freely, simplicially.

② Construct finite-dim contractible $X \cong \prod_n X'$ with G-action } uses finite-index

③ Prove G-action is free + simplicial } uses torsion-free

How we construct X:

As a set, $X = \mathrm{Hom}_H(G, X')$ (i.e. maps $f: G \to X'$ s.t. $f(hg) \mapsto h \cdot f(g)$ $\forall h \in H$, $g \in G$)

If we pick coset representatives $g_1, \dots, g_n$, then equivalent to independently choosing $f(g_1), \dots, f(g_n)$. So $X \cong \prod_n X'$

Eg: $G = \mathbb{Z}$, $H = n\mathbb{Z}$. Coset reps $0, 1, \dots, n-1$.

$X' = \mathbb{R}$. $X = \mathbb{R}^n$ $\quad (a_0, \dots, a_{n-1})$, $a_0 = f(0)$, $a_{n-1} = f(n-1)$ $\qquad f(k) = f(n+k)$

$\mathbb{Z} \curvearrowright \mathbb{R}^n$ by $k \cdot (a_0, \dots, a_{n-1}) \mapsto (a_{0+k}, \dots, a_{n-1+k})$

Rmk: This is analogous to algebraically constructing "coinduced modules".

$SL_n\mathbb{Z}$ has torsion, so Serre's Theorem doesn't directly apply, but we can apply it to finite-index torsion-free subgroups of $SL_n\mathbb{Z}$:

$$\begin{tikzcd}
 & SL_n\mathbb{Z} \arrow[dl, dash, "m"'] \arrow[dr, dash, "\ell"] \arrow[dd, dash, "\leq m\ell"] & \\
H \arrow[dr, dash, "\leq \ell"'] & & K \arrow[dl, dash, "\leq m"] \\
 & H\cap K &
\end{tikzcd}$$

Serre's Thm $\Rightarrow$ $\boxed{cd(H) = cd(H\cap K) = cd(K)}$.

This allows us to make the following defn:

Defn: The virtual cohomological dim (vcd) of a group G is the coh. dim of any finite-index torsion-free subgp of G.

Fact: $vcd(SL_n\mathbb{Z}) = \binom{n}{2}$

Other known groups: $vcd(Mod_g) = 4g-5$ ($Mod_g \curvearrowright Teich_g$)

$vcd(Sp_{2g}(\mathbb{Z})) = g^2$

$vcd(Out(F_n)) = 2n-2$

Ques: How does knowing vcd actually help with $H^i(SL_n\mathbb{Z}; -)$?

Rational coefficients help:

Defn: The rational cohomological dimension of G is

$$cd_{\mathbb{Q}}(G) := \max\{n \mid H^n(\underset{\uparrow \atop K(G,1)}{G}; V) \quad V: \mathbb{Q}\text{-vector space}\}$$

Theorem: $H \subset G$ finite index. Then $cd_{\mathbb{Q}}(H) = cd_{\mathbb{Q}}(G)$

Rmk:
- Note absence of torsion-free assumption.
- $cd_{\mathbb{Q}}(G) \leq cd(G)$

Cor: $cd_{\mathbb{Q}}(G) \leq vcd(G)$

In fact, we also have the following:

Thm: $\tilde{X}$ contractible, $G \curvearrowright \tilde{X}$ simplicially w/ finite stabilizers. Then

$$H_*(G;\mathbb{Q}) = H_*(\tilde{X}/G;\mathbb{Q})$$

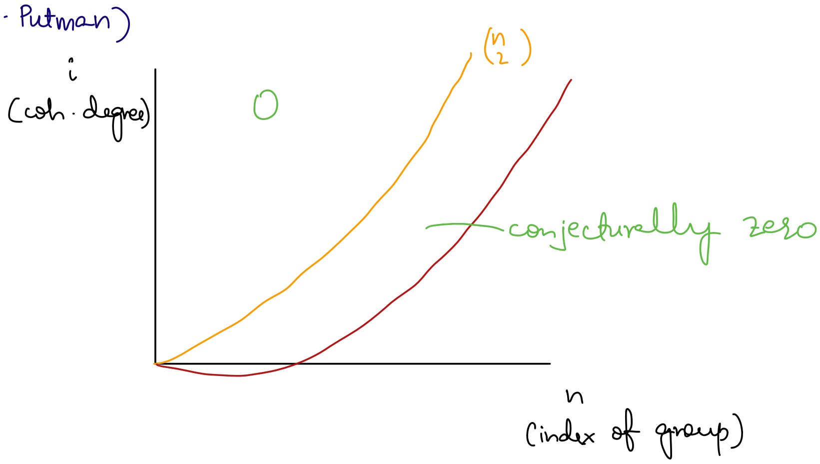
(Note: in the case of $G = SL_n\mathbb{Z}$, $\tilde{X} = SL_n\mathbb{R}/SO(n)$)

III Borel-Serre duality

We'll shift our attention now towards $\mathbb{Q}$-coefficients.

We know $cd_{\mathbb{Q}}(SL_n\mathbb{Z}) \leq vcd(SL_n\mathbb{Z}) = \binom{n}{2}$

Conjecture: (Church-Farb-Putman) $\quad H^{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q}) = 0$ for $n >> i$



We have a duality result that helps us compute these higher dim. cohomologies:

Borel-Serre duality (for $SL_n\mathbb{Z}$): $\quad H^{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q}) \cong H_i(SL_n\mathbb{Z};\mathbb{Q}\otimes_{\mathbb{Z}} D)$

Here $D := St_n\mathbb{Q}$ ("Steinberg module")

$= \tilde{H}^{n-2}(T_n;\mathbb{Z})$

↑ Tits building

Similar result for (virtual) duality groups, in general.
In that case, $D = H^{vcd(G)}(G; \mathbb{Z}G)$

So finding higher dim cohomology $\leftrightarrow$ low dim. homology, which can be done algebraically through constructing partial resolutions

Progress on Conjecture: $H^{\binom{n}{2}}(SL_n\mathbb{Z}; \mathbb{Q}) = 0$ for $n \geq 2$ (Lee-Szczarba)

$H^{\binom{n}{2}-1}(SL_n\mathbb{Z}; \mathbb{Q}) = 0$ for $n \geq 3$ (Church-Farb-Putman 2016)

$H^{\binom{n}{2}-2}(SL_n\mathbb{Z}; \mathbb{Q}) = 0$ for $n \geq 3$ (Brück, Miller, Patzt, Sroka, Wilson)