I (Co)homology of a Group

G - group

Fact : $\exists$ a topological space, called the $K(G,1)$ - space X,

s.t.

$$\pi_1(X) \cong G$$

and
$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array} \quad \text{univ. cover contractible}$$

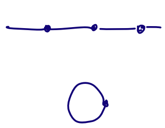
X is unique upto homotopy

Defn / Fact :

- The (co)homologies of this $K(G,1)$ space X are the (co)homologies of the group G.

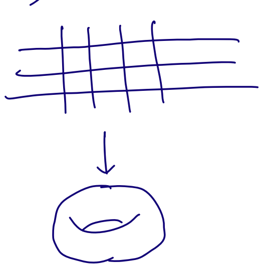
Examples

1) $G = \mathbb{Z}$



$$\begin{array}{c} \mathbb{R} \\ \downarrow \\ S^1 \end{array} \quad - \ K(\mathbb{Z},1)\text{-space}$$

2) $G = \mathbb{Z} \times \mathbb{Z}$



$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1 \end{array}$$
$$- \ K(\mathbb{Z}^2, 1)$$

Fact: We can construct a $K(G,1)$ s.t. $\tilde{X}$ is a simplicial complex, with $G \curvearrowright \tilde{X}$ simplicial action.

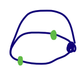
Remark : There is an algebraic version of this definition, using free or projective resolutions over the ring $\mathbb{Z}G$. The algebraic & topological definitions turn out to be equivalent. The algebraic version allows us to define (co)homology with coefficients in any $\mathbb{Z}G$-module M.

Eg (3) $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccccc} S^2 & \subset & S^3 & \subset & S^4 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}P^2 & \subset & \mathbb{R}P^3 & \subset & \mathbb{R}P^4 \end{array}$$

not contractible $\to S^2$

$e^0 \cup e^1 \cup e^2$ $\qquad$ $e^0 \cup e^1 \cup e^2 \cup e^3$ $\qquad$ $e^0 \cup e^1 \cup e^2 \cup e^3 \cup e^4$



$C_2 \to C_1 \to C_0 \to 0$

$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z} \to 0$

$H_1 = \mathbb{Z}/2\mathbb{Z}$

$H_2 = 0$

$C_3 \to C_2 \to C_1 \to C_0 \to 0$

$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$H_1 = \mathbb{Z}/2\mathbb{Z}$

$H_2 = 0$

$H_3 = \mathbb{Z}$

$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z} \to 0$

$H_1 = \mathbb{Z}/2\mathbb{Z}$

$H_3 = \mathbb{Z}/2\mathbb{Z}$

$H_2, H_4 = 0$

Form union of all S^n, $\mathbb{R}P^n$:

$$\begin{array}{c} S^\infty \\ \downarrow \\ \mathbb{R}P^\infty \end{array}$$

$S^\infty \leftarrow$ contractible!

$$H_n(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

We'll be interested in the family of groups $\{SL_n\mathbb{Z}\}$, and seeing what happens to cohomology in "higher dimensions"

(II) Cohomological Dimension

Defn: The cohomological dimension of a group G, denoted $cd(G)$, is
$$\max\{n \mid H^n(X; M) \neq 0\}$$
X: $K(G,1)$ $\qquad$ M: $\mathbb{Z}G$-module

One way to get a bound on $cd(G)$ is to get a bound on $\dim X$.

Eg: $cd(\mathbb{Z}) = 1$, $cd(\mathbb{Z}^2) = 2$,
$cd(\mathbb{Z}_2) = \infty$

Fact: For all cases we'll consider, the dim. $cd(G)$ can be attained by a $K(G,1)$ space.

Useful fact: If $H \subset G$ subgroup, then
$$cd(H) \leq cd(G)$$

"**Proof**": Find a $K(G,1)$ $\begin{array}{c}\tilde{X}\\ \downarrow \\ X\end{array}$ s.t. $\dim \tilde{X} = cd(G)$.

Then G has a covering space action on $\tilde{X}$, and $X = \tilde{X}/G$.

Restrict to H: $\tilde{X}/H$ is a $K(G,1)$ for H, of dim $cd(G)$.

Remark: The above fact is almost immediate using the algebraic defn of group cohomology.

Fact: $\mathbb{Z}_m$ has $cd = \infty$.

So: If G has torsion, it has infinite cohomological dim.

Let's turn our attention now to the group $SL_n\mathbb{Z}$...

III The Group $SL_n\mathbb{Z}$

→ Want to study cohomology of $SL_n\mathbb{Z}$ and its coh. dim.

→ Turns out $SL_n\mathbb{Z}$ has cd $\binom{n}{2}$ ... kinda ...

→ **Plan**: Find a contractible space on which $SL_n\mathbb{Z}$ acts by a covering space action
  → free
  → properly discontinuous

$SL_n\mathbb{Z} \curvearrowright SL_n\mathbb{R}/SO_n \cong Sym_n/\{\text{scalar multiples}\}$

dim: $(n^2-1)-\binom{n}{2}$

$= \binom{n+1}{2}-1$

+ve def sym. $n\times n$ matrices

<u>n = 2 case:</u>

$SL_2\mathbb{Z} \curvearrowright SL_2\mathbb{R}/SO_2 \cong \mathbb{H}^2$

$SL_2\mathbb{R} \curvearrowright \mathbb{H}^2$ by fractional linear maps, transitively

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$\mathrm{Stab}(i) = SO_2$

<u>Problems</u>: (1) Action not free (eg. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ fixes i)

(2) $SL_n\mathbb{Z}$ has torsion (eg. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$)

(3) $\mathrm{cd}\, SL_n\mathbb{Z} = \binom{n}{2}$??

<u>Solution</u>: (1) Stabilizers are <u>finite</u>

(2) $SL_n\mathbb{Z}$ has torsion-free subgroups of <u>finite</u> index

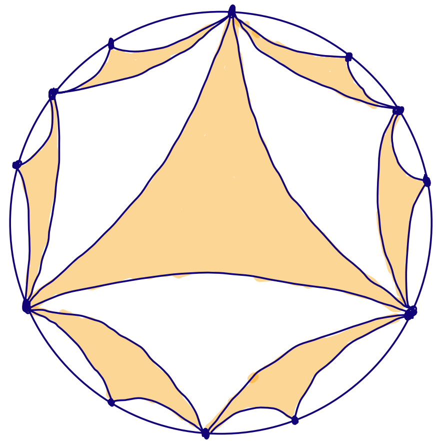
(3) vcd (virtual coh. dim) of $SL_n\mathbb{Z} = \binom{n}{2}$

} not a coincidence!

<u>Back to n = 2</u>: $\binom{n}{2} = \binom{2}{2} = 1$. What we have: $\dim(\mathbb{H}^2) = 2$.

Rest of today: See how to cut down dim to 1.

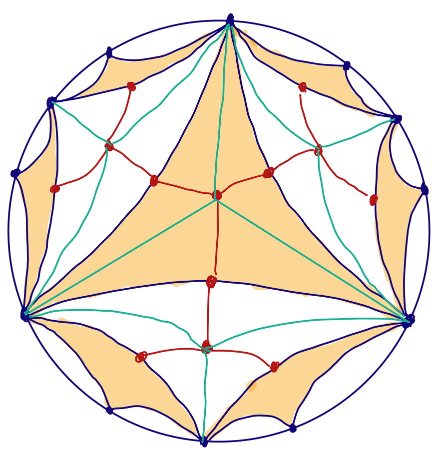
We'll use the disk model of $\mathbb{H}^2$. Consider the following tiling of the disk by triangles:



Here we started with the large center triangle, and obtained more triangles by successively reflecting each vertex about the opposite side. Alternate triangles are shaded here in orange.

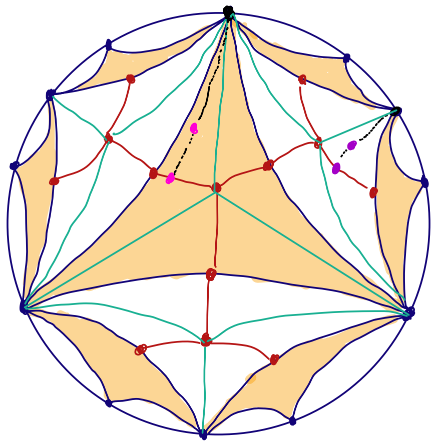
One can check that these triangles are preserved by the $SL_2\mathbb{Z}$-action (it's easier to do this using the corresponding picture in $\mathbb{H}^2$, and using the fact that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2\mathbb{Z}$ and preserve the triangles).

We want to somehow cut this down to a 1-dim graph. Barycentrically subdivide:



Remove all simplices with a vertex on the boundary (what's left is the 1-dim maroon graph).

Now we can deformation retract our original simplicial complex onto the maroon graph as follows:



Can also check that $SL_2\mathbb{Z}$ acts simplicially on the maroon graph.