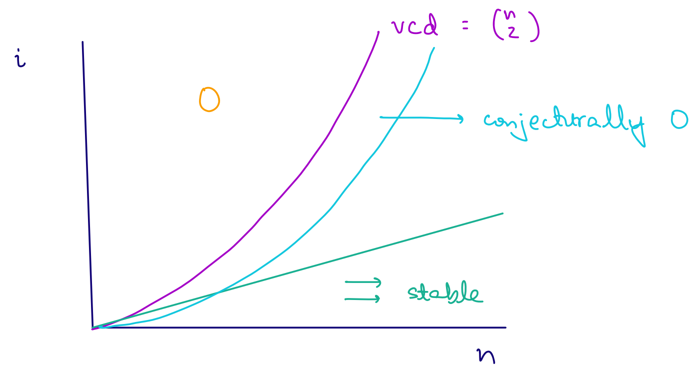
(Rational) Duality Groups and $H^*(SL_n\mathbb{Z};\mathbb{Q})$

Ⓘ Motivation: Calculating $H^*(SL_n\mathbb{Z};\mathbb{Q})$

$H^i(SL_n\mathbb{Z};\mathbb{Q})$



Ⓘⓘ Group (Co)homology

Γ: Group, M: module over $\mathbb{Z}\Gamma$

We can define (co)homology groups $H^*(\Gamma;M)$ and $H_*(\Gamma;M)$

(Above we're viewing $\mathbb{Q}$ as a $\mathbb{Z}\Gamma$-module with trivial Γ-action.)

Algebraically: Take a free (more generally, projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

→ For $H_*(\Gamma;M)$, apply $-\otimes_{\mathbb{Z}\Gamma} M$ and take homologies.

Fact: $H_0(\Gamma;M) \cong M_\Gamma = M/\langle m-\gamma m \mid \gamma\in\Gamma\rangle$

→ For $H^*(\Gamma;M)$, apply $\mathrm{Hom}_{\mathbb{Z}\Gamma}(-,M)$ and take cohomologies.

Fact: $H^0(\Gamma;M) \cong M^\Gamma = \langle m \mid \gamma m = m \ \forall \gamma\in\Gamma\rangle$

Topologically: Suppose $\exists$ contractible CW-complex X with $\Gamma \curvearrowright X$
- free
- cellular

Then
$$\boxed{\begin{aligned} H_*(\Gamma;M) &\cong H_*(X/\Gamma;M) \\ H^*(\Gamma;M) &\cong H^*(X/\Gamma;M)\end{aligned}}$$

Upshot: If X is a "nice" space, we can use it to study algebraic properties of Γ, such as cohomological dimension.

III Cohomological dimension

Defn: • (Cohomological dimension)

$cd(\Gamma) := \max\{n \mid H^n(\Gamma; M) \neq 0 \text{ for a } \mathbb{Z}\Gamma\text{-module } M\}$

• (Rational cohomological dimension)

$cd_{\mathbb{Q}}(\Gamma) := \max\{n \mid H^n(\Gamma; V) \neq 0 \text{ for a } \mathbb{Q}\Gamma\text{-module } V\}$

Note: $cd_{\mathbb{Q}}(\Gamma) \leq cd(\Gamma)$

If $\exists$ contractible X s.t. $SL_n\mathbb{Z} \curvearrowright X$ freely + cellularly s.t. $\dim X = k$, then $cd(\Gamma) \leq k$.

Defn: Symmetric Space

$SL_n\mathbb{Z} \curvearrowright SL_n\mathbb{R}/SO(n) = X_n$ "symmetric space"

$g \cdot A = gAg^T$

$\dim X_n = \binom{n+1}{2} - 1$

In fact, we can further "trim down" X_n to a space Y_n of $\dim \binom{n}{2}$.

- X_n (and Y_n) is contractible
- action is not free! Problem: finite stabilizers

But if we pass to a torsion-free subgroup $\Gamma' \subset SL_n\mathbb{Z}$, then $\Gamma' \curvearrowright X_n$ is free.

We can still salvage information from $SL_n\mathbb{Z} \curvearrowright X_n$, by turning to rational coefficients.

Fact: [Serre] If $[\Gamma : \Gamma'] < \infty$, then $cd_{\mathbb{Q}}(\Gamma') = cd_{\mathbb{Q}}(\Gamma)$

(The same holds for $cd(\Gamma)$ if Γ is torsion-free, which $SL_n\mathbb{Z}$ is not)

Fact: $SL_n\mathbb{Z}$ has torsion-free subgroups of finite index (Eg: ker of $SL_n\mathbb{Z} \twoheadrightarrow SL_n(\mathbb{Z}/p)$)

So from the above, we have $\boxed{cd_{\mathbb{Q}}(SL_n\mathbb{Z}) \leq \binom{n}{2}}$ (in fact, equality holds)

Thus $H^i(SL_n\mathbb{Z}; \mathbb{Q}) = 0$ for $i > \binom{n}{2}$

Q: What about $i \leq \binom{n}{2}$?

A: We can use a duality property

IV (Rational) Duality Groups

Defn: Let R be a ring (eg. $\mathbb{Z}$ or $\mathbb{Q}$). We say Γ is a "duality group over R" if $\exists$ integer k and $R\Gamma$-module D s.t. we have

$$H^i(\Gamma; M) \cong H_{k-i}(\Gamma; \underset{\substack{\uparrow \\ \text{dualising} \\ \text{module}}}{D} \otimes_R M)$$

$\forall$ $R\Gamma$-modules M

(Here $\Gamma \curvearrowright D \otimes_R M$ diagonally, i.e. $\gamma(d \otimes m) = d\gamma^{-1} \otimes \gamma m$.
In particular, $(D \otimes_R M)_\Gamma \cong D \otimes_{R\Gamma} M$)

Rmk: By plugging $M = R\Gamma$, we see that $D \cong H^k(\Gamma; R\Gamma)$:

$$D = D \otimes_{R\Gamma} R\Gamma = (D \otimes_R R\Gamma)_\Gamma = H_0(\Gamma; D \otimes_R R\Gamma) \cong H^k(\Gamma; R\Gamma)$$

Thm: $SL_n\mathbb{Z}$ is a duality group over $\mathbb{Q}$.
Thus $H^i(SL_n\mathbb{Z}; \mathbb{Q}) \cong H_{\binom{n}{2}-i}(SL_n\mathbb{Z}; D \otimes_\mathbb{Q} \mathbb{Q})$ "Poincare duality with a twisting of the coefficients"

Rmk: • We can think of this as analogous to "Poincare duality with a twisting of the coefficients"

• Thus finding high dimensional cohomology of $SL_n\mathbb{Z}$ translates to finding low dimensional homology - albeit at the cost of a more complicated coefficient module — which we can hope to do by constructing partial resolutions.

Goal: • Prove $SL_n\mathbb{Z}$ is a rational duality group
• Get a concrete description of the dualising module D. (Turns out to be $St_n\mathbb{Q} \otimes \mathbb{Q}$ "Steinberg Module")

For both these answers, we'll look at the action on symmetric space
$$SL_n\mathbb{Z} \curvearrowright X_n = SL_n\mathbb{R}/SO(n)$$

Because $SL_n\mathbb{Z} \curvearrowright X_n$ is not free, we'd like to be able to pass to a torsion-free subgroup.

Fact: If $[\Gamma : \Gamma'] < \infty$, then Γ is a duality gp. over $\mathbb{Q}$ iff Γ' is
(same holds for duality gp over $\mathbb{Z}$, if Γ is torsion-free)

Fact: If Γ is a duality gp over $\mathbb{Z}$, then Γ is a duality gp over $\mathbb{Q}$ w/ dualising module $D \otimes_\mathbb{Z} \mathbb{Q}$

So our goal now is to prove any finite-index torsion-free $\Gamma' \subset SL_n\mathbb{Z}$ is a duality gp over $\mathbb{Z}$, and to find its dualising module.

<u>Thm</u> [Bieri Eckmann] Suppose Γ is of type FP [i.e. $\mathbb{Z}$ has a finite length, finite type proj. res. over $\mathbb{Z}\Gamma$]

TFAE:

(1) Γ is a duality gp over $\mathbb{Z}$

(2) $H^i(\Gamma; \mathbb{Z}\Gamma) = 0 \ \forall\ i \neq k$

$H^k(\Gamma; \mathbb{Z}\Gamma)$ is torsion-free

<u>(Glimpse of Proof)</u>: Suppose $P_k \to P_{k-1} \to \dots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is a finite length+type res.

(2) + type FP $\Rightarrow$ $H^k(\Gamma; \mathbb{Z}\Gamma) \leftarrow Hom(P_k, \mathbb{Z}\Gamma) \leftarrow \dots \leftarrow Hom(P_0, \mathbb{Z}\Gamma) \leftarrow 0$ is a projective resolution.

This allows us to calculate $H^i(\Gamma; M)$ and $H_{k-i}(\Gamma; D \otimes M)$ from the same resolution.

<u>Updated Goal</u>: $\Gamma' \subset SL_n\mathbb{Z}$ fin. index, torsion-free

- $H^i(\Gamma'; \mathbb{Z}\Gamma') = 0 \ \forall\ i \neq \binom{n}{2}$ $\longrightarrow$ We want to relate these with untwisted, integral (co)homology of a topological space

$H^{\binom{n}{2}}(\Gamma'; \mathbb{Z}\Gamma')$ torsion-free

<u>Fact</u>: If X is contractible and X/Γ' cpct, then
$$H^i(\Gamma'; \mathbb{Z}\Gamma') \cong H^i_c(X; \mathbb{Z})$$

<u>Problem</u>: X_n/Γ' is not compact in our case

<u>Solution</u>: Borel-Serre Compactification

<u>Thm</u> [Borel-Serre] There exists a compactification $X_n \subset \overline{X}_n$ s.t.:

- $X_n \hookrightarrow \overline{X}_n$ is a homotopy equivalence
- $\overline{X}_n/\Gamma$ is compact
- $\overline{X}_n$ is a topological (not smooth) manifold of dim $= \binom{n+1}{2} - 1$
- $\partial\overline{X}_n \simeq \mathcal{T}_n\mathbb{Q}$ "Tits building" - is a simplicial complex, $\simeq \bigvee S^{n-2}$

Thus, we now have:

$$H^i(\Gamma'; \mathbb{Z}\Gamma') \underset{\substack{\uparrow \\ \overline{X}_n/\Gamma' \\ \text{is cpct}}}{=} H^i_c(\overline{X}_n; \mathbb{Z}) \overset{\substack{\text{Poincare duality} \\ \text{for } \overline{X}_n}}{=} H_{\binom{n+1}{2}-i-1}(\overline{X}_n, \partial\overline{X}_n; \mathbb{Z})$$

$$\overset{\substack{\text{LES of pair,} \\ \overline{X}_n \simeq \{*\}}}{=} H_{\binom{n+1}{2}-i-2}(\partial\overline{X}_n; \mathbb{Z})$$

$$\overset{\partial\overline{X}_n \simeq \mathcal{T}_n\mathbb{Q}}{=} H_{\binom{n+1}{2}-i-2}(\mathcal{T}_n\mathbb{Q}; \mathbb{Z})$$

Since $\mathcal{T}_n\mathbb{Q} \simeq \bigvee S^{n-2}$, H_* is concentrated in deg $n-2$ (and is free abelian for deg $n-2$)

i.e., precisely when:

$$\binom{n+1}{2} - i - 2 = n-2$$

$$\iff \boxed{i = \binom{n}{2}}$$

Thus:
- $SL_n\mathbb{Z}$ is a duality gp over $\mathbb{Q}$
- Dualising module $= H_{n-2}(\mathcal{T}_n\mathbb{Q};\mathbb{Z}) \otimes \mathbb{Q} = \underbrace{St_n\mathbb{Q}}_{\text{"Steinberg module"}} \otimes \mathbb{Q}$

(Time Permitting:)

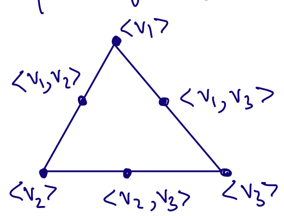
V The Tits Building $\mathcal{T}_n(\mathbb{Q})$

Vertices $\longleftrightarrow$ $0 \subsetneq V \subsetneq \mathbb{Q}^n$ proper subspaces

p-simplices $\longleftrightarrow$ Flags of subspaces
$$0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq \mathbb{Q}^n$$

$\dim \mathcal{T}_n\mathbb{Q} = n-2$

Eg: Suppose $\langle v_1, v_2, v_3\rangle = \mathbb{Q}^3$. Then the following is a subcomplex of $\mathcal{T}_3\mathbb{Q}$:



"apartment classes"

Thm: [Solomon-Tits] $\mathcal{T}_n\mathbb{Q} \simeq \bigvee S^{n-2}$

$St_n\mathbb{Q} = H_{n-2}(\mathcal{T}_n\mathbb{Q};\mathbb{Z})$ is generated by apartment classes.

Note: $SL_n\mathbb{Z} \curvearrowright \mathcal{T}_n\mathbb{Q}$, so $St_n\mathbb{Q}\otimes\mathbb{Q}$ is a $\mathbb{Q}[SL_n\mathbb{Z}]$-module.

A picture to see how $\partial \bar{X}_2 \simeq \tau_2 \mathbb{Q}$:

Note : $\tau_2 \mathbb{Q} = \mathbb{Q} \cup \{\infty\}$ (discrete set of vertices)

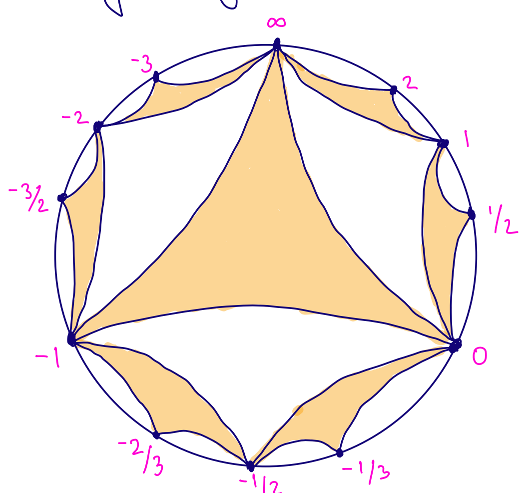
$SL_2 \mathbb{Z} \curvearrowright SL_2 \mathbb{R} / SO_2 \cong \mathbb{H}^2$

$SL_2 \mathbb{R} \curvearrowright \mathbb{H}^2$ by fractional linear maps, transitively

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$\mathrm{Stab}(i) = SO_2$

Disk Model of $\mathbb{H}^2$: Consider the following tiling of the disk by triangles :



Here we started with the large center triangle, and obtained more triangles by successively reflecting each vertex about the opposite side. Alternate triangles are shaded here in orange.

Rmk : These triangles are preserved by the $SL_2\mathbb{Z}$ - action
(Using generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $SL_2\mathbb{Z}$)

Note : The "corners" on the boundary circle are not in $\mathbb{H}^2$, but we can add them in without changing the homotopy type.
The resulting quotient $\bar{X}_2 / SL_2 \mathbb{Z}$ is also compact.

The added points exactly correspond to $\mathbb{Q} \cup \{\infty\}$, so
$\boxed{\partial \bar{X}_2 \simeq \tau_2 \mathbb{Q}}$