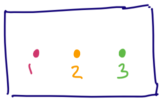
"The Homology of the Little Disks Operad" – Dev Sinha : An Informal Summary

First Pass - An Overview

Goal: To understand $\mathrm{Conf}_n(\mathbb{R}^d)$, specifically its H_* and H^*.

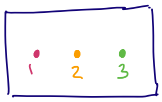
$\mathrm{Conf}_n(\mathbb{R}^d) := \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}^d,\ x_i \neq x_j\}$



A point in $\mathrm{Conf}_3 \mathbb{R}^2$

We'll focus on concrete combinatorial descriptions of H_* and H^*.
We'll describe the $d=2$ case first, which captures the heart of the picture, and then describe the general case.

Ⓘ $H_*(\mathrm{Conf}_n(\mathbb{R}^2))$ and Trees



A point in $\mathrm{Conf}_3 \mathbb{R}^2$

- $H_1(\mathrm{Conf}_n(\mathbb{R}^2)) \rightsquigarrow$ "1-dim structure"
  view loops as "particle dances"

Eg:

View this as a map

$S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$

So get induced map

$H_1(S^1) \longrightarrow H_1(\mathrm{Conf}_n \mathbb{R}^2)$
$\quad \mathbin{\rotatebox[origin=c]{90}{$\cong$}}$
$\quad \mathbb{Z}$

We can use this strategy to study higher-degree H_k.

- $H_2(\mathrm{Conf}_n \mathbb{R}^2) \rightsquigarrow$ "2-dim structure"

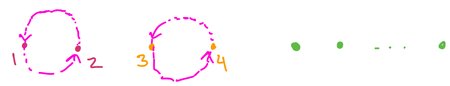
  We can construct maps

  $S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$

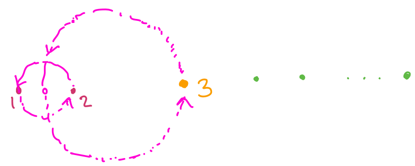
  And look at the (images of) induced map

  $H_2(S^1 \times S^1) \longrightarrow H_2(\mathrm{Conf}_n \mathbb{R}^2)$
  $\quad \mathbin{\rotatebox[origin=c]{90}{$\cong$}}$
  $\quad \mathbb{Z}$

Eg:



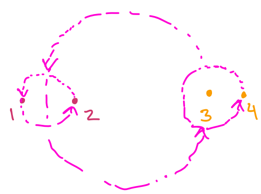
Eg:



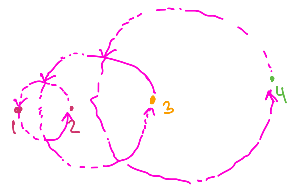
• $H_3(\mathrm{Conf}_n \mathbb{R}^2)$

$S^1 \times S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$

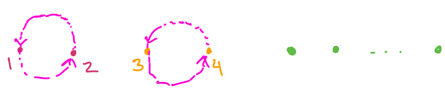
Eg:

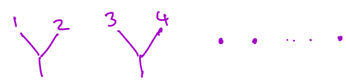


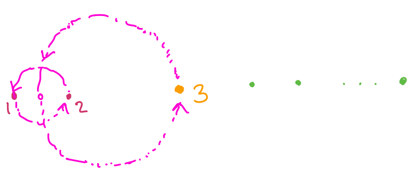
Eg:

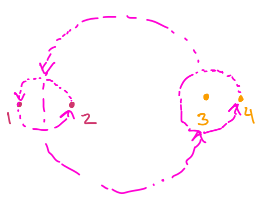


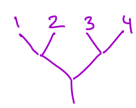
So we're constructing homology classes via "orbiting planet systems".
We can represent these orbiting systems using trees:

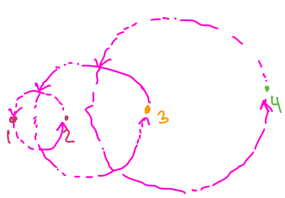


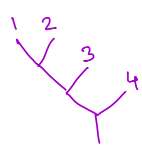


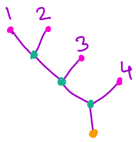








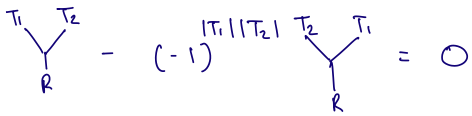




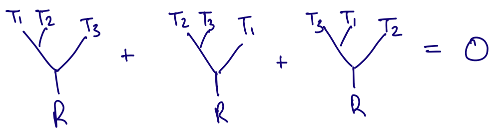
- internal vertices $\rightsquigarrow$ # internal vertices $|T|$ gives the homology degree
- root vertex
- leaves $\rightsquigarrow$ # leaves gives the no. of particles.

So given a (labelled) forest, we have an associated homology class.

<u>Note</u>:
- We have so far only described <u>some</u> homology gp. elements. We can get others by, for eg, taking linear combinations of them.
- Turns out, the above described classes <u>generate</u> all the homology. But they have some relations between them, i.e. they are not a basis.

- 

(Anti-Symmetry)

- 

(Jacobi)

T_i, R are subtrees

<u>Rmk</u>: In general, it is usually easier to describe generators of configuration spaces than their relations.

(II) <u>Cohomology & Graphs</u>

How do we start thinking about H^*?

<u>Just for the purposes of this talk</u>, $H^k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{Z})$

(In general, we always have a natural surjection $H^k \to \mathrm{Hom}(H_k, \mathbb{Z})$, which is an iso if, for eg, all H_k's are torsion-free)

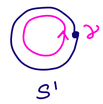
So for eg. $H^1(S^1) \cong \mathrm{Hom}(\underset{\underset{\mathbb{Z}}{\text{SII}}}{H_1(S^1)}, \mathbb{Z}) \cong \mathbb{Z}$

To study $H^1(\mathrm{Conf}_n \mathbb{R}^2)$, we'll construct maps

$$\mathrm{Conf}_n \mathbb{R}^2 \to S^1$$

to get induced

$$\underset{\underset{\mathbb{Z}}{\text{SII}}}{H^1(S^1)} \to H^1(\mathrm{Conf}_n \mathbb{R}^2)$$

- $\omega \in H^1(S^1)$ 

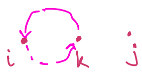
ν : generates $H_1(S^1)$
Have $\omega \in H^1(S^1)$ sending $\nu \mapsto 1$

Analogous cohomology class in $H^1(\mathrm{Conf}_n \mathbb{R}^2)$:

$\alpha_{ij} \in H^1(\mathrm{Conf}_n \mathbb{R}^2)$

α_{ij} "extracts the motion of i wrt j"

α_{ij} () = 1

 α_{ij} () = 0

Rigorously, we have a map

$$a_{ij} : \mathrm{Conf}_n \mathbb{R}^2 \longrightarrow S^1$$
$$(x_1, .., x_n) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

ω : Generator of

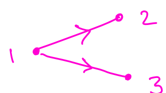
$$\boxed{\alpha_{ij} := a_{ij}^*(\omega)}$$

Represent α_{ij} by a (directed) graph 

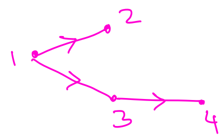
So,
- we've described some elements of $H^1(\mathrm{Conf}_n \mathbb{R}^n)$
- turns out, they also generate H^1
- we understand their action on H_1, as $H_1 \longrightarrow \mathbb{Z}$

→ What about higher H^k?

Using the cohomology cup product, we can get some higher-degree cohomology classes.
We'll represent these as graphs too.

Eg: $\alpha_{12}\, \alpha_{13} \in H^2$ 

$\alpha_{34}\alpha_{12}\alpha_{13} \in H^3$



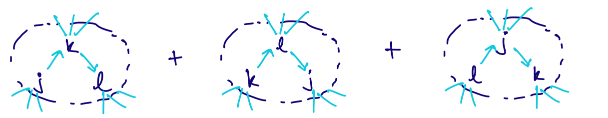
→ Implicit in these graphs is an ordering σ of the edges, to record the order of multiplication of the α_{ij}'s.

Note: # edges $|E(G)|$ ⇝ degree of cohomology $|E(G)|$

Relations:

- If G_1, G_2 differ in reversal of k arrows and edge reordering by σ,

  $G_1 - (\operatorname{sgn}\sigma) G_2 = 0$ (Arrow-Reversing)

- 

  $= 0$ (Arnold)

  "dual to Jacobi for H_*"

Proposition: Any H^* class represented by a graph with cycles is 0.

Pf Sketch: We can use the Arnold identity inductively to reduce to graphs with shorter cycles. Thus we can reduce to graphs with cycle length 2, i.e. more than one edge b/w 2 vertices.

Since $\omega^2 = 0$ in $H^*(S^1)$, we have $\alpha_{ij}^2 = 0$. Thus graphs with multiple edges b/w two vertices are zero.

→ Thus we can take all our graphs to be acyclic

→ But how can we understand these graphs as $\operatorname{Hom}(H_k, \mathbb{Z})$?

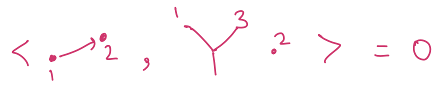
III Graph-Tree Pairing

So far, we have a way of associating cohomology classes to directed labelled graphs (with an ordering on edges)

To understand how they act on H_k, we need to unpack how the cup product works.

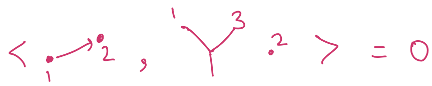
But it turns out, there is a combinatorial rule that captures this.

This should, for eg, give us: 



Here's how we define the general pairing $\langle G, T \rangle$:

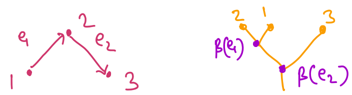
① If there's an edge $i \to j$ in G s.t. there is no path b/w i & j in T, then $\langle G, T \rangle = 0$

Eg: 

② Otherwise, define

$\beta_{G,T} : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$

$i \to j \mapsto$ lowest vertex of path from i to j.

Eg: 

$$\langle G, T \rangle = \begin{cases} 0 & \text{if } \beta_{G,T} \text{ not a bijection} \\ \pm 1 & \text{o/w} \end{cases}$$

depends on σ and order of leaf labels on T

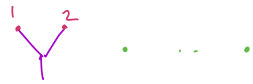
IV The General d Case

$d > 2$

What changes? → Loops collapse (for eg in $\mathbb{R}^3$, we have enough dimensions to collapse the loop )

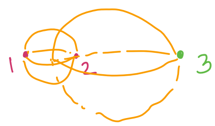
→ Don't have any homology until degree $d-1$

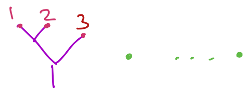
Eg $d = 3$



$S^2 \to \mathrm{Conf}_n \mathbb{R}^3$

$H_2(S^2) \to H_2(\mathrm{Conf}_n \mathbb{R}^3)$





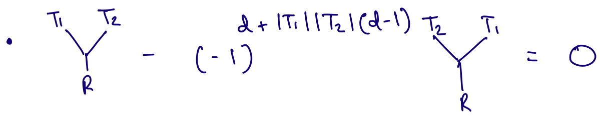
$S^2 \times S^2 \to \mathrm{Conf}_n \mathbb{R}^3$

$H_4(S^2 \times S^2) \to H_4(\mathrm{Conf}_n \mathbb{R}^3)$

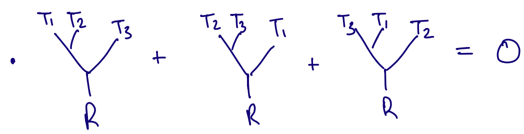
→ Can still use trees & forests to represent these classes

→ A tree T gives a degree $|T|(d-1)$ homology class

Relations:



(Anti-Symmetry)

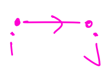


(Jacobi)

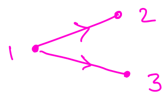
For cohomology, we define maps

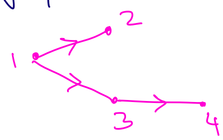
$a_{ij} : \mathrm{Conf}_n \mathbb{R}^d \to S^{d-1}$

$\boxed{\alpha_{ij} := a_{ij}^*(\omega)}$ $\quad \omega \in H^{d-1}(S^{d-1})$

We represent α_{ij} with a directed edge 

And products of the α_{ij} using (directed) graphs.

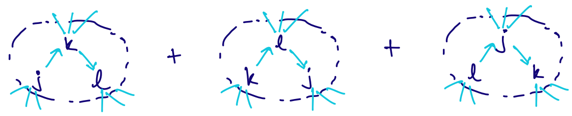
$\alpha_{12}\alpha_{13}$ 

$\alpha_{34}\alpha_{12}\alpha_{13}$ 

Relations :

- If G_1, G_2 differ in reversal of k arrows and edge reordering by σ,

$$G_1 - (-1)^{kd}(\operatorname{sgn}\sigma)^{d-1} G_2 = 0 \qquad \text{(Arrow-Reversing)}$$

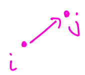
- 

$= 0$ (Arnold)

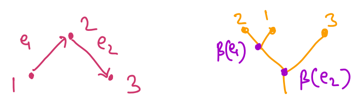
Similar to the $d=2$ case, $\alpha_{ij}^2 = 0$ and we can take all graphs to be acyclic.

The Graph-Tree pairing is also similar :

$$\beta_{G,T} : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$$



$\mapsto$ lowest vertex of path from i to j.

Eg: 

$$\langle G, T\rangle = \begin{cases} 0 & \text{if } \beta_{G,T} \text{ not a bijection} \\ \pm 1 & \text{o/w} \end{cases}$$

depends on σ, d, and order of leaf labels on T

Second Pass – Some Proof Outlines and Ideas

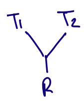
(I) Pf of Anti-Symmetry and Arrow-Reversal Relations

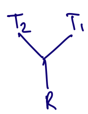
Recall that we defined certain homology classes in $\mathrm{Conf}_n\mathbb{R}^d$ associated to trees T by embedding a product of spheres $(S^{d-1})^{\times |T|}$ in $\mathrm{Conf}_n\mathbb{R}^d$. Let the embedded submanifold be P_T.

To define these homology classes integrally, we need to orient P_T. We can orient P_T by parameterizing it through a map from $(S^{d-1})^{\times |T|}$, and using the orientation on the torus.

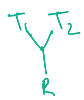
The relations of anti-symmetry on H_* and arrow-reversing on H^* arise from how certain reparameterizations affect this orientation.

(i) Anti-Symmetry:



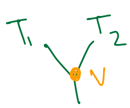
$- \; (-1)^{d + |T_1||T_2|(d-1)}$ 

$= 0$

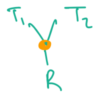
Both 

and 

define the same submanifold, but with different parameterizations.



$$: \underbrace{S^{d-1} \times \cdots \times S^{d-1}}_{\text{correspond to internal vertices of } T_1} \times \underbrace{S^{d-1}}_{\text{corresponds to } v} \times \underbrace{S^{d-1} \times \cdots \times S^{d-1}}_{\text{correspond to internal vertices of } T_2} \longrightarrow \mathrm{Conf}_n\mathbb{R}^d$$

The swapping of T_1 and T_2 corresponds to the antipodal map on the factor of S^{d-1} corresponding to the vertex v  — which gives the $(-1)^d$ sign — and in shuffling all the factors of S^{d-1} corresponding to the internal vertices of T_2 to the front — which gives the $(-1)^{|T_1||T_2|(d-1)}$ sign.

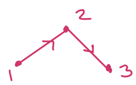
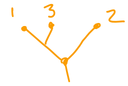
② Arrow - Reversing

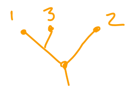
- If G_1, G_2 differ in reversal of k arrows and edge reordering by σ,

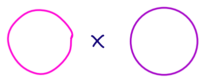
$$G_1 - (-1)^{kd} (\operatorname{sgn} \sigma)^{d-1} G_2 = 0$$

The $(-1)^{kd}$ arises from the antipodal map on each S^{d-1} corresponding to the k reversed edges, and $(\operatorname{sgn} \sigma)^{d-1}$ arises from the reordering of the order in which the $\alpha_{ij} \in H^{d-1}$ corresponding to each edge have been multiplied.

II Why the graph-tree pairing rule works : A proof in pictures

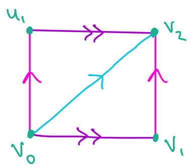
Eg : $\alpha_{12}\alpha_{23}$ =  , 

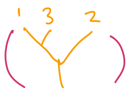
Recall :  is obtained from the image of the map

F :  $\longrightarrow$ 

$S^1 \times S^1$

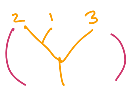
Give a Δ-complex structure to $S^1 \times S^1$:



Then $\alpha_{12}\alpha_{23}\Big($  $\Big) = \alpha_{12}\alpha_{23}\big(F[v_0 u_1 v_2] + F[v_0 v_1 v_2]\big)$

$$= \underbrace{\alpha_{12}(F[v_0u_1])}_{0}\,\alpha_{23}(F[u_1,u_2]) + \alpha_{12}(F[v_0v_1])\,\underbrace{\alpha_{23}(F[v_1v_2])}_{0}$$

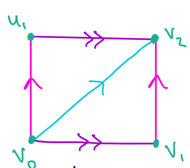
$$= 0$$

On the other hand, for $\alpha_{12}\alpha_{23}\Big($  $\Big)$, the analogous computation yields :

$\alpha_{12}\alpha_{23}\Big($  $\Big) = \alpha_{12}\alpha_{23}\big(F[v_0 u_1 v_2] + F[v_0 v_1 v_2]\big)$

$$= \underbrace{\alpha_{12}(F[v_0u_1])}_{1}\,\underbrace{\alpha_{23}(F[u_1u_2])}_{1} + \underbrace{\alpha_{12}(F[v_0v_1])}_{0}\,\alpha_{23}(F[v_1v_2]) = 1$$

where

F :  $\longrightarrow$ 

$S^1 \times S^1$

In general, we can give $(S^1)^k$ a Δ-complex structure by partitioning $[0,1]^k$ into $k!$ k-simplices of the form $[v_0 v_1 \ldots v_k]$, where each $v_i v_{i+1}$ is an edge of $[0,1]^k$.

Thus for $\omega \in H^k$, $h \in H_k$, $\omega(h)$ is a sum of terms of the form $\alpha_{i_0 i_1}([v_0, v_1]) \, \alpha_{i_1 i_2}([v_1, v_2]) \ldots \alpha_{i_{k-1} i_k}([v_{k-1}\, v_k])$.

The image of each edge $[v_j\, v_{j+1}]$ is a single S^1-orbit in the homology class. Note that orbits correspond to internal vertices of the associated tree

Thus, a given term is ± 1 iff $[v_j\, v_{j+1}]$ maps to an orbit s.t. particles i_j and i_{j+1} are on different components of that orbit.

This can happen iff the pairs (i_j, i_{j+1}) — which correspond to the edges of the graph G — can be put in bijection with the orbits — corresponding to internal vertices of T — s.t. each pair i_j, i_{j+1} is on different components of the associated orbit. This is exactly what the combinatorial rule said.

All other terms will be forced to be 0, and thus $\omega(h) \neq 0$ iff a bijection of the above form exists.

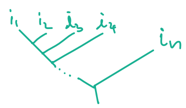
(III) Determining $H_*(\mathrm{Conf}_n \mathbb{R}^d)$ and $H^*(\mathrm{Conf}_n \mathbb{R}^d)$

An outline :

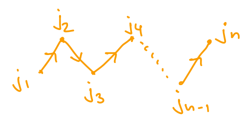
→ We describe some H_* classes using forests, and some H^* classes using graphs

→ We have certain relations among these forests and graphs.

→ Using these relations, we can deduce that all forests and graphs by those whose components consist of "tall trees" and "long graphs".



A tall tree
(here i_1 is minimal among the i_k)



A long graph
(here j_1 is minimal among the j_k)

→ The graph-tree pairing on tall trees and long graphs is perfect, in the sense that it is 1 iff $(i_1, i_2, \dots, i_n) = (j_1, \dots, j_n)$, and 0 otherwise.

This lets us deduce linear independence of these H_* and H^* elements, and thus gives us a lower bound on the ranks of H_* and H^*.

→ A spectral sequence argument on the fibration

$$\bigvee_{n-1} S^{d-1} \to \mathrm{Conf}_n \mathbb{R}^d \to \mathrm{Conf}_{n-1} \mathbb{R}^d$$

gives an upper bound on the rank of $H_* \mathrm{Conf}_n \mathbb{R}^d$ (of $H_*(\bigvee_{n-1} S^{d-1}) \otimes H_* \mathrm{Conf}_{n-1} \mathbb{R}^d$)

This upper bound agrees with the rank of forests spanned by tall trees, and thus such forests form a basis for all the homology.

Third Pass - More Proofs

(I) Formalising the "orbit system" submanifolds

Given a labelled tree T, we want to write a map $(S^{d-1})^{\times |T|} \longrightarrow \mathrm{Conf}_n \mathbb{R}^d$ that parameterises the "orbit system submanifold" P_T.

We do this as follows:

$$S^{d-1} \times S^{d-1} \times \ldots \times S^{d-1} \longrightarrow \mathrm{Conf}_n \mathbb{R}^d$$

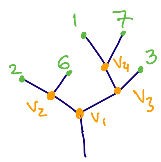
$$(u_{v_1}, u_{v_2}, \ldots\ldots, u_{v_{|T|}}) \longmapsto (x_1, x_2, \ldots, x_n)$$

where

$$x_i = \sum_{\substack{v_j \text{ below} \\ \text{leaf } i}} \pm \varepsilon^{h_j} u_{v_j}$$

here:

- $\varepsilon < 1/3$ is fixed
- h_j is the height of vertex v_j, i.e. the no. of edges b/w v_j and the root
- The sum is taken over all vertices v_j which lie on the path from leaf i to the root.
- The $\pm$ sign is $+1$ if the path from i to the root goes through the left edge of v_j and -1 if it goes through the right edge.

Eg: $T =$ 

$h(v_1) = 1$, $h(v_2) = 2 = h(v_3)$, $h(v_4) = 3$

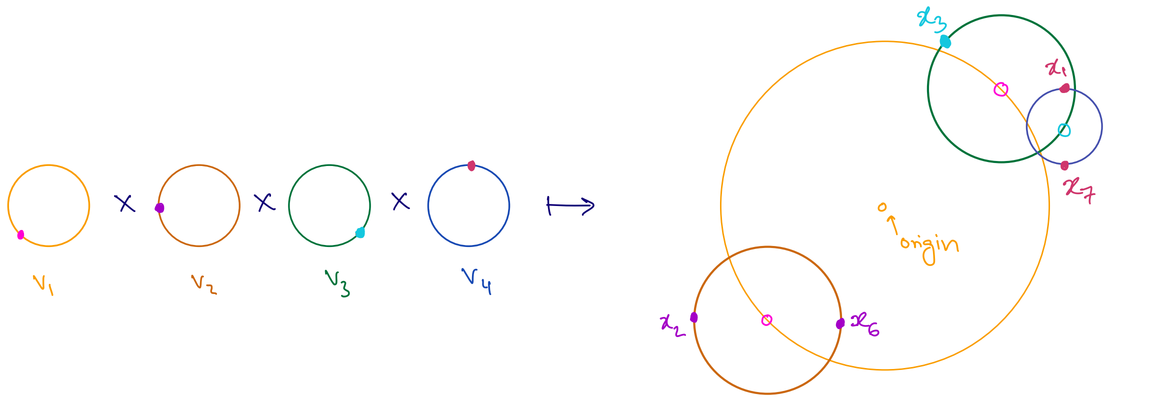
$$x_1 = -\varepsilon u_{v_1} + \varepsilon^2 u_{v_3} + \varepsilon^3 u_{v_4}$$

$$x_2 = \varepsilon u_{v_1} + \varepsilon^2 u_{v_2}$$

$$x_3 = -\varepsilon u_{v_1} - \varepsilon^2 u_{v_3}$$

$$x_6 = \varepsilon u_{v_1} - \varepsilon^2 u_{v_2}$$

$$x_7 = -\varepsilon u_{v_1} + \varepsilon^2 u_{v_3} - \varepsilon^3 u_{v_4}$$



radius of orange circle $= \varepsilon$

radius of brown circle $= \varepsilon^2$

radius of green circle $= \varepsilon^2$

radius of blue circle $= \varepsilon^3$

II Proof of the Arnold Identity

A precursor : determining $H^{d-1}(\mathrm{Conf}_n \mathbb{R}^d) \cong \mathbb{Z}^{\binom{n}{2}}$

Sketch of proof : Induct on n.

For $n=1$, $\mathrm{Conf}_1 \mathbb{R}^d \cong \mathbb{R}^d$, so $H^{d-1} = 0$.

We have a (split) fibration

$$\bigvee_n S^{d-1} \to \mathrm{Conf}_{n+1} \mathbb{R}^d \to \mathrm{Conf}_n \mathbb{R}^d$$

The action of $\pi_1(\mathrm{Conf}_n \mathbb{R}^d)$ on the fibre is trivial, and we can use the Leray-Serre spectral sequence to deduce that

$$\begin{aligned}\operatorname{rank} H_{d-1}(\mathrm{Conf}_{n+1} \mathbb{R}^d) &\leq \operatorname{rank}\big(H_0(\mathrm{Conf}_n \mathbb{R}^d) \otimes H_{d-1}(\textstyle\bigvee_n S^{d-1}) \\ &\qquad \oplus H_{d-1}(\mathrm{Conf}_n \mathbb{R}^d) \otimes H_0(\textstyle\bigvee_n S^{d-1})\big) \\ &\leq n + \binom{n}{2} = \binom{n+1}{2}\end{aligned}$$

Thus $\operatorname{rank} H^{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d) = \operatorname{rank} H_{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d) \leq \binom{n+1}{2}$

Now, consider the $\binom{n+1}{2}$ elements $\alpha_{ij} = \underset{i}{\bullet}\!\longrightarrow\!\underset{j}{\bullet}$ of $H^{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d)$, and the $\overset{i\ \ j}{Y}$ in $H_{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d)$.

The pairing on these elements is perfect, i.e.

$\langle \underset{i}{\bullet}\!\longrightarrow\!\underset{j}{\bullet} , \overset{k\ \ \ell}{Y} \rangle = 1$ if $(i,j) = (k,\ell)$ and 0 o/w.

Thus all these elements must be independent.

Thus $\binom{n+1}{2} \leq \operatorname{rank} H^{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d) \leq \binom{n+1}{2}$

This implies that $H^{d-1}(\mathrm{Conf}_{n+1}\mathbb{R}^d)$ is spanned by the α_{ij}. In fact, the $i \bullet\!\!\rightarrow\!\! \bullet j$ and $\overset{k\ \ \ell}{Y}$ are dual bases for H^{d-1} and H_{d-1}.

(Alternatively, could take LES of π_k's on the fibration. Since the fibration splits, the LES splits into short exact sequences.

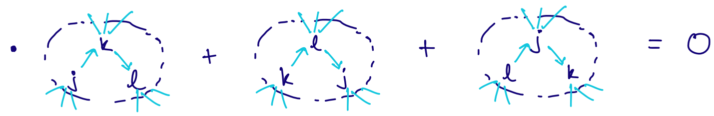
Can inductively deduce that $\mathrm{Conf}_n\mathbb{R}^d$ is $(d-1)$-conn, and that $\pi_{d-1}(\mathrm{Conf}_n\mathbb{R}^d)$ is free abelian. Moreover,

$$\operatorname{rank} \pi_{d-1}(\mathrm{Conf}_n\mathbb{R}^d) = \operatorname{rank} \pi_{d-1}\mathrm{Conf}_{n-1}\mathbb{R}^d + \operatorname{rank} \pi_{d-1} \bigvee_{n-1} S^{d-1}$$
$$= \binom{n-1}{2} + n-1 \quad = \binom{n}{2}$$

Then Hurewicz $\Rightarrow$ $H_{d-1}\mathrm{Conf}_n\mathbb{R}^d \cong \pi_{d-1}\mathrm{Conf}_n\mathbb{R}^d \cong \mathbb{Z}^{\binom{n}{2}}$

Note that this argument only works for $d > 2$.)

The Arnold Identity



It suffices to prove when there are no edges incident on j, k, l other than the two involved in the identity.
Thus it suffices to show that $\alpha_{jk}\alpha_{kl} + \alpha_{kl}\alpha_{lj} + \alpha_{lj}\alpha_{jk} = 0$.
Assume that $\{j, k, l\} = \{1, 2, 3\}$, and that we're working in $\mathrm{Conf}_3 \mathbb{R}^d$.

Pf Idea : To express this as a cup product that is demonstably 0.
We'll use the fact that cup products are dual to intersections.

Proof : Since $\mathrm{Conf}_3 \mathbb{R}^d$ is a manifold, its cohomology is Poincare-Lefschetz dual to its locally finite homology.
Consider the submanifold of $(x_1, x_2, x_3) \in \mathrm{Conf}_3 \mathbb{R}^d$ s.t. x_1, x_2, x_3 are collinear. This submanifold has 3 components. Let Col_i denote the one in which x_i is in the middle.

Col_i is a submanifold of codimension $d-1$
(Suppose $x_i = (x_i^1, \ldots, x_i^d)$. Then x_1, x_2, x_3 being collinear means that $\dfrac{x_1^j - x_2^j}{x_1^j - x_3^j} = \dfrac{x_1^1 - x_2^1}{x_1^1 - x_3^1}$ for all $2 \le j \le d$. This gives us $d-1$ constraints)
When oriented, Col_i gives a locally finite homology class in codim $d-1$.
Thus the Poincare-Lefschetz dual of Col_i is in $H^{d-1}(\mathrm{Conf}_3 \mathbb{R}^d)$. Thus it is a linear combination of $\alpha_{12}, \alpha_{23}, \alpha_{31}$. We can find what this combination is by intersecting Col_i with various  .

(*)
(see explanation for why this is true at the end of the proof)

__Note__: Col_i can only intersect ${}^{i}\mathrm{Y}^{j}$ and ${}^{i}\mathrm{Y}^{k}$ and does so at exactly one point each.

Moreover, these intersections differ in sign by -1, coming from orientation-reversing of the line on which the 3 points lie.

Thus the dual of Col_i is $\pm(a_{ij} - a_{ik})$.

Since Col_1 and Col_2 are disjoint, their duals cup product to 0.

Thus:
$$\begin{aligned} 0 &= (a_{12} - a_{13})(a_{23} - a_{21}) \\ &= a_{12}a_{23} - a_{12}a_{21} - a_{13}a_{23} + a_{13}a_{21} \\ &= a_{12}a_{23} - 0 - (-1)^{d+(d-1)} a_{23}a_{31} + (-1)^{2d} a_{31}a_{12} \\ &= a_{12}a_{23} + a_{23}a_{31} + a_{31}a_{12} \end{aligned}$$

Explanation of $(*)$

Here's a general fact:

Let M be an m-dim manifold, and $\omega \in H^k(M)$.
Let $\sigma \in H_k(M)$ and let $\omega_L \in H_{m-k}(M)$ be the Poincare-Lefschetz dual of ω.
Then,
$$\omega(\sigma) = (\text{sign of}) \ \sigma \cap \omega_L$$
(In our case, $\omega = \mathrm{Col}_i$, $\sigma = {}^{i}\mathrm{Y}^{j}$)

__Here's why__: Let $[M]$ denote the fundamental class of M.
Let $\sigma_L \in H^{m-k}(M)$ be the dual of σ.

Then,
$$\omega(\sigma) = \omega([M] \frown \sigma_L) \overset{\text{cap-cup relation}}{=\!=} (\sigma_L \smile \omega)([M]) \overset{\text{cup-intersection duality}}{=\!=} (\mathrm{sgn})\ \sigma \cap \omega_L$$

Extra : A Topological Pf (generalising the above idea) of the H^*-relations for hypesplane complements

Setup : $H_1, H_2, \ldots, H_k$ are (complex) hyperplanes in $\mathbb{C}^n$

$a_1, a_2, \ldots, a_k$ are normal vectors to these planes

$\alpha_i : \mathbb{C}^n \to \mathbb{C}$ given by $\alpha_i = \langle a_i, - \rangle$

(Thus $H_i = \ker \alpha_i$)

Let $\omega_i \in H^1(\mathbb{C}^n \setminus H_1 \cup \ldots \cup H_k)$ be the "winding no. around H_i"

i.e. ω_i is the pullback of a gen. of $H^1(\mathbb{C}-\{0\})$ under the map $\alpha_i : \mathbb{C}^n \setminus H_1 \cup \ldots \cup H_k \to \mathbb{C}-\{0\}$

Suppose the α_i's are linearly dependent. We want to show that

$$\sum_{i=1}^{k} (-1)^{i-1} \omega_1 \cup \ldots \cup \hat{\omega}_i \cup \ldots \cup \omega_k = 0 \qquad (*)$$

Proof : (I) Sketch :

Suppose $\alpha_k = c_1\alpha_1 + \ldots + c_{k-1}\alpha_{k-1}$

We can replace α_i with $c_i\alpha_i$, thus getting:

$$\alpha_k = \alpha_1 + \cdots + \alpha_{k-1} \iff 1 = \frac{\alpha_1}{\alpha_k} + \cdots + \frac{\alpha_{k-1}}{\alpha_k}$$

Let M_i denote the submanifold of $\mathbb{C}^n \setminus H_1 \cup \ldots \cup H_k$ defined by $\frac{\alpha_i}{\alpha_k} \in \mathbb{R}_{<0}$.

Then M_i is a codim 1 oriented submfld. Also, since $\sum \frac{\alpha_i}{\alpha_k} = 1$, we have $M_1 \cap M_2 \cap \ldots \cap M_k = \phi$.

Let $\sigma_i \in H^1$ be the Poincare-Lefschetz dual of M_i.

The cup product - intersection duality implies that

$$\sigma_1 \sigma_2 \ldots \sigma_{k-1} = 0$$

If we show that $\sigma_i = \pm(\omega_k - \omega_i)$, then this implies

$$(\omega_k - \omega_1)(\omega_k - \omega_2)\dots(\omega_k - \omega_{k-1}) = 0 \qquad (**)$$

Note that the left side of $(**)$ simplifies precisely to $(*)$.

II Steps to show that $\sigma_i = \pm(\omega_k - \omega_i)$

1. Prove that $\omega_1, \dots, \omega_k$ form a basis for $H^1(\mathbb{C}^n \setminus H_1 \cup \dots \cup H_k)$ via a Mayer-Vietoris argument. Thus each σ_i is a linear combination of the ω_j.

2. Construct loops $\gamma_1, \dots, \gamma_k$ that are the algebraic duals to $\omega_1, \dots, \omega_k$, i.e. $\omega_i(\gamma_j) = \delta_{ij}$.

3. Show that M_i only intersects the loops γ_i and γ_k, each at a single point, and in opposite orientations.

   This would imply that $\sigma_i(\gamma_j) = \begin{cases} \pm 1 & j = i \\ -\sigma_i(\gamma_i) & j = k \\ 0 & o/w \end{cases}$ ——— $(*)$

   Thus $\sigma_i = \pm(\omega_k - \omega_i)$

Here's an explanation of why $(*)$ holds:

Let M be an m-dim manifold, and $\omega \in H^k(M)$.
Let $\sigma \in H_k(M)$ and let $\omega_L \in H_{m-k}(M)$ be the Poincare-Lefschetz dual of ω.
Then,

$$\omega(\sigma) = (\text{sign of})\ \sigma \cap \omega_L$$

Here's why: Let $[M]$ denote the fundamental class of M.
Let $\sigma_L \in H^{m-k}(M)$ be the dual of σ.

Then,

$$\omega(\sigma) = \omega([M] \frown \sigma_L) \overset{\text{cap-cup relation}}{=\!=} (\sigma_L \smile \omega)([M])$$

$$\overset{\text{cup-intersection duality}}{=\!=} (\text{sgn})\, \sigma \cap \omega_L$$

III Execution of Steps 1, 2, 3

① $\omega_1, \omega_2, \dots, \omega_k$ form a basis for H^1.

Induction on k

First, two observations:

(A) If V is a codim d subspace of $\mathbb{C}^n$, then

$$\mathbb{C}^n - V \underset{\text{deformation retract}}{\simeq} \mathbb{C}^d - \{0\} \simeq S^{2d-1}$$

In particular, if $d \geq 2$, then H^1 and H^2 of $\mathbb{C}^n - V$ are 0.

(B) Now if $V_1, \dots, V_k$ are all codim $d \geq 2$, then we can inductively use Mayer-Vietoris to see that

$$H^1(\mathbb{C}^n \setminus V_1 \cup \dots \cup V_k) = 0$$

$$H^2(\mathbb{C}^n \setminus V_1 \cup \dots \cup V_k) = 0$$

Returning to the proof :

- Base Case $k=1$: $\mathbb{C}^n - H_1 \simeq \mathbb{C} - \{0\}$ by A
  so ω_1 is a basis for $H^1(\mathbb{C}^n - H_1)$

- Induction Step : suppose $\omega_1, \dots, \omega_{k-1}$ form a basis of $H^1(\mathbb{C}^n \setminus H_1 \cup \dots \cup H_{k-1})$

Apply Mayer - Vietoris to $\mathbb{C}^n \setminus H_1 \cup \dots \cup H_{k-1}$ and $\mathbb{C}^n \setminus H_k$.

Their intersection is $\mathbb{C}^n \setminus H_1 \cup \dots \cup H_k$ and union is $\mathbb{C}^n \setminus (H_k \cap H_1) \cup \dots \cup (H_k \cap H_{k-1})$.

By B , H^1 and H^2 of the latter vanish, so Mayer - Vietoris $\Rightarrow$

$$H^1(\mathbb{C}^n \setminus H_1 \cup \dots \cup H_k)) \cong H^1(\mathbb{C}^n \setminus H_1 \cup \dots \cup H_{k-1}) \oplus H^1(\mathbb{C}^n \setminus H_k)$$

and is freely gen. by $\omega_1, \dots, \omega_{k-1}, \omega_k$.

2 Constructing algebraic duals γ_i to ω_i

Recall that $\alpha_i : \mathbb{C}^n - (H_1 \cup \dots \cup H_k) \to \mathbb{C} - \{0\}$. suppose ω generates $H^1(\mathbb{C} - \{0\})$.

Then by defn $\omega_i = \alpha_i^*(\omega)$. i.e., if γ is a loop, then $\omega_i(\gamma) = \omega(\alpha_i(\gamma))$.

So the loops $\gamma_1, \dots, \gamma_k$ we want to construct must satisfy $\omega(\alpha_i(\gamma_j)) = \delta_{ij}$.

Let $\beta_1, \dots, \beta_\ell \in (\mathbb{C}^n)^* = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$ be s.t. ℓ is a minimal so that $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell$ span $(\mathbb{C}^n)^*$.

We can construct loops, i.e. 1-dim submflds, by specifying what values they should take under $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_\ell$.
This has to be subject to $\alpha_1 + \dots + \alpha_{k-1} = \alpha_k$.

Let γ_1 be the submfld on which
$$\alpha_1 \in S^1$$
$$\alpha_2, \alpha_3, \dots, \alpha_{k-1} = 2$$
$$\alpha_k = \alpha_1 + \dots + \alpha_{k-1}$$
$$\beta_i = 0$$

Imp: Orient γ with a chosen orientation of S^1

Similarly construct $\gamma_2, \dots, \gamma_{k-1}$.

Let γ_k be given by:
$$\alpha_k \in S^1$$
$$\alpha_2, \alpha_3, \dots, \alpha_{k-1} = -2$$
$$\alpha_1 = \alpha_k - \alpha_2 - \dots - \alpha_{k-1}$$
$$\beta_i = 0$$

It is straightforward to check that $\omega_i(\gamma_j) = \delta_{ij}$

③ <u>M_1 intersects only γ_1 and γ_k, and at opposite orientations</u>

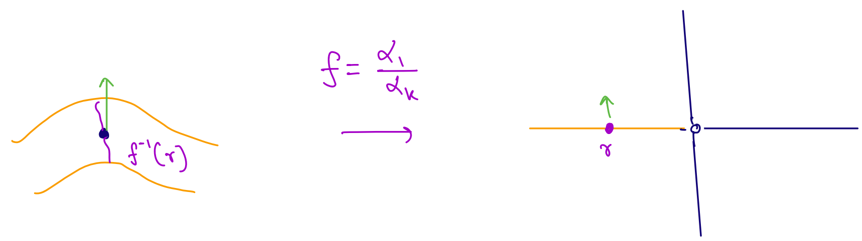
Recall that M_1 is defined by $\frac{\alpha_1}{\alpha_k} \in \mathbb{R}_{<0}$

The only point on γ_1 (resp. γ_k) satisfying this is the one where α_1 (resp. α_k) is -1.

It remains to show that these intersections happen at opposite orientations.

First, we need to fix an orientation on M_1. Since M_1 is a codim 1 submfld of the orientable mfld $\mathbb{C}^n \setminus H_1 \cup \dots \cup H_k$, it suffices to pick a unit normal vector for M_1.

Locally, M_1 looks like this:



In $\mathbb{C}^n \setminus H_1 \cup \dots \cup H_k$ $\qquad\qquad\qquad\qquad$ $\mathbb{C} - \{0\}$

Pick the direction of the normal vector to correspond to the "upward" direction in $\mathbb{C} - \{0\}$.

Now, consider the iso. $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ that acts on $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$ as follows: $\quad \Phi(x) = y$ so that

$$\beta_i(y) = \beta_i(x)$$
$$\alpha_1(y) = \alpha_k(x), \quad \alpha_k(y) = \alpha_1(x)$$
$$\alpha_i(y) = -\alpha_i(x), \quad i = 2, \dots, k-1$$

Thus Φ "swaps" α_1 and α_k.

Note that $\Phi(\gamma_1) = \gamma_k$, preserving orientations. And $\Phi(M_1) = M_1$.

If we consider the composition $\frac{\alpha_1}{\alpha_k} \circ \Phi : M_1 \to \mathbb{R}_{<0}$, this is equivalent to composing $\Phi : M_1 \to \mathbb{R}_{<0}$ with $x \to 1/x : \mathbb{R}_{<0} \to \mathbb{R}_{<0}$.

This reverses the orientation of $\mathbb{R}_{<0}$, and from this we can deduce that Φ reverses the orientation of M_1.

Thus $\gamma_k \cap M_1 = \Phi(\gamma_1 \cap M_1)$, and the intersections happen at opposite orientations.