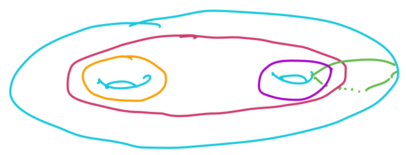
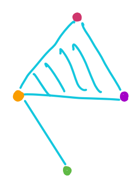
(I) The Curve Complex $C(S^b_{g,n})$

k-simplices $\leftrightarrow$ (k+1) disjoint isotopy classes of simple closed curves

("curve system")





Remarks:

→ not allowing curves $\simeq *$ or peripheral to a puncture or boundary component
(these curves would be disjoint from everything and hence cone off the curve complex)

→ $C(S^b_{g,n}) \cong C(S_{g,n+b})$

Theorem [Harer '86]

$$C(S_{g,n}) \simeq \bigvee_{\infty} S^m$$

$$m = \begin{cases} n-4 & \text{if } g=0 \\ 2g-2 & \text{if } g \geq 1,\ n=0 \\ 2g-3+n & \text{if } g \geq 1,\ n \neq 0 \end{cases}$$

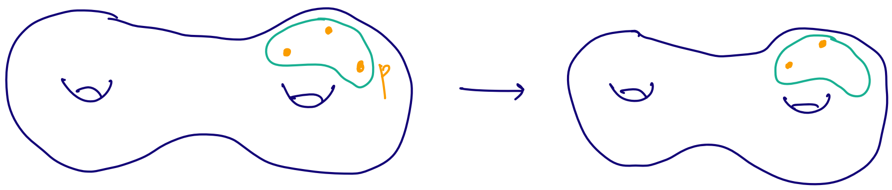
$\dim(C(S_{g,n})) = 3g+n-4$ (because we need $3g+n-3$ curves for a pants decomposition)

Proof consists of two main parts : (1) Inducting on n
(2) Base Cases of $n = 0, 1$

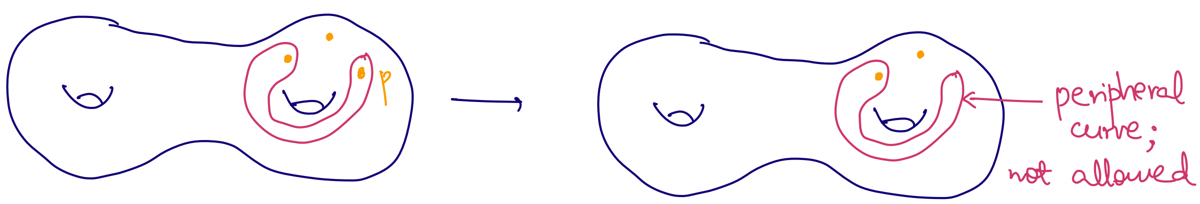
II Recursive Structure of $C(S_{g,n})$: Inducting on n

Prop^n: If $n \geq 2$, then $C(S_{g,n}) \simeq \underbrace{A_{g,n}}_{\text{discrete set}} * C(S_{g,n-1})$

Idea: Want to define a "forget a point" map $C(S_{g,n}) \to C(S_{g,n-1})$



But we can't define this on all of $C(S_{g,n})$

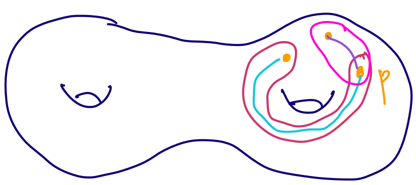


Let $X_{g,n} \subset C(S_{g,n})$: subcomplex spanned by "good curves"

(Thus we have a "forget a pt" map $X_{g,n} \xrightarrow{f} C(S_{g,n-1})$)

$A_p \subset C(S_{g,n})$: subcomplex spanned by "bad curves"

Note: bad curves are exactly the ones peripheral to an arc joining p and another puncture



No two such curves can be disjoint, and so A_p is a discrete set.

Thus, $\boxed{C(S_{g,n}) \subset A_p * X_{g,n}}$

We'll show that : $\boxed{C(S_{g,n}) \simeq \mathcal{A}_p * X_{g,n} \simeq \mathcal{A}_p * C(S_{g,n-1})}$

We will need the following Lemma, whose proof can be found in Section V :

<u>Lemma:</u> X any simplicial complex
A any discrete set
$Y \subset A * X$ and $A, X \subseteq Y$ s.t.
$\forall a \in A$, $lk_Y(a) \hookrightarrow X$ is a htpy equiv.
Then $Y \hookrightarrow A * X$ is a htpy equiv.

For us, $Y = C(S_{g,n})$, $A = \mathcal{A}_p$, $X = X_{g,n}$

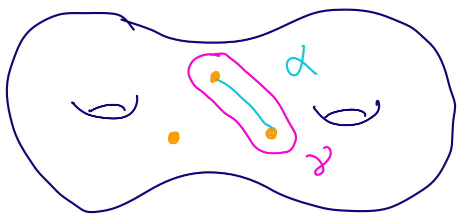
For $\gamma \in \mathcal{A}_p$, consider

$$lk_{C(S_{g,n})} \gamma \xhookrightarrow{\iota} X_{g,n} \xrightarrow{f} C(S_{g,n-1})$$

We'll show that : • $f \circ \iota$ is a simplicial iso
• ι_* is surjective on all π_k

This will imply that both ι, f are htpy equiv. (as they induce isos on π_k) , and using the Lemma, we will get

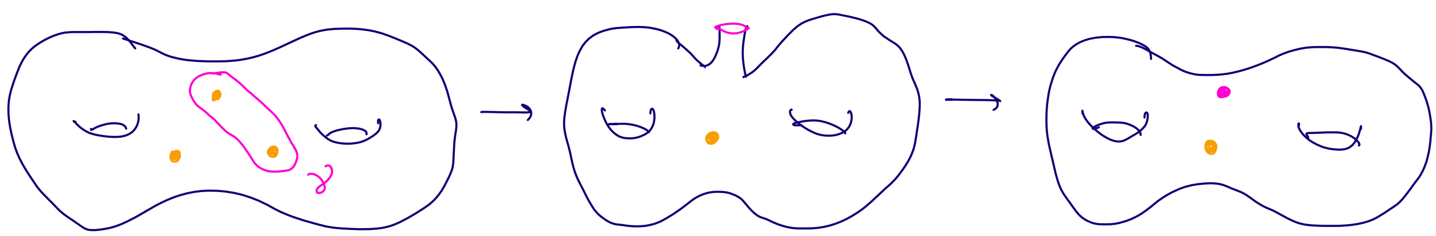
$\boxed{C(S_{g,n}) \simeq \mathcal{A}_p * X_{g,n} \simeq \mathcal{A}_p * C(S_{g,n-1})}$



Let α be the arc that γ is peripheral to.

Now, the link of γ is spanned by all curves disjoint to γ. All such curve systems must avoid passing through the interior of the disk bounded by γ.

We can identify such curve systems as curve systems on $S'_{g,n-2}$, which is iso to the curve complex on $S_{g,n-1}$.



Thus $lk_{C(S_{g,n})} \cong C(S_{g,n-1})$. Note that the composition $lk_{C(S_{g,n})}\gamma \xrightarrow{\iota} X_{g,n} \xrightarrow{f} C(S_{g,n-1})$ realises this simplicial iso.

Thus ι_* is injective on π_k's.

For surjectivity, we use an idea of Hatcher, called "Hatcher flow".

Here's the idea:

Given any map $\Psi: S^k \to X_{g,n}$, we want to homotope it so its image lies in $lk_{C(S_{g,n})}\gamma$.

Fix a simplicial structure on S^k, and assume Ψ is simplicial.

Let $v_1, v_2, \dots, v_r$: vertices of S^k

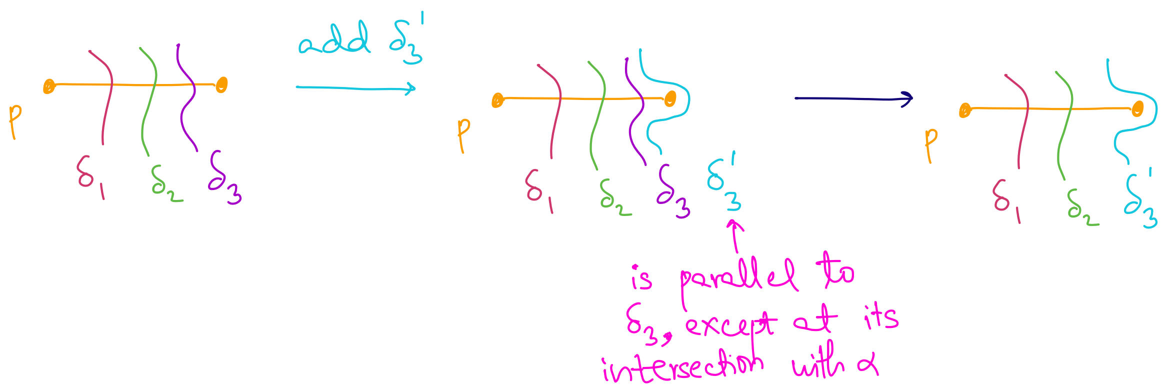
$\delta_1, \delta_2, \dots, \delta_r$: curves representing $\Psi(v_i)$

We need to homotope Ψ so that $\delta_i \cap \alpha = \phi$

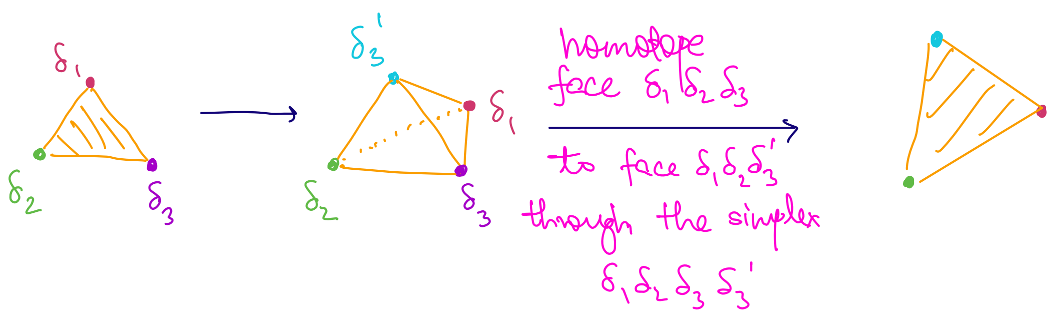
arc defining γ

Here's the idea of Hatcher flow:

On $S_{g,n}$:



In $C(S_{g,n})$:



Continue in this way until all curves have been homotoped off of α.

Thus, we have proved that for $n \geq 2$, $C(S_{g,n}) \simeq A_p * C(S_{g,n-1})$

So if $C(S_{g,n-1}) \simeq \vee S^m$, then $C(S_{g,n})$ will be $\simeq \vee S^{m+1}$.

It now remains to deal with the base cases of $n = 0, 1$.

Rmk: When $n=1$ we do have a "forget a pt" map $f: C(S_{g,1}) \to C(S_{g,0})$, but no bad curves. In fact, in this case f is a htpy equiv, as we will see in Section IV.

III Connectivity of $C(S_{g,1})$

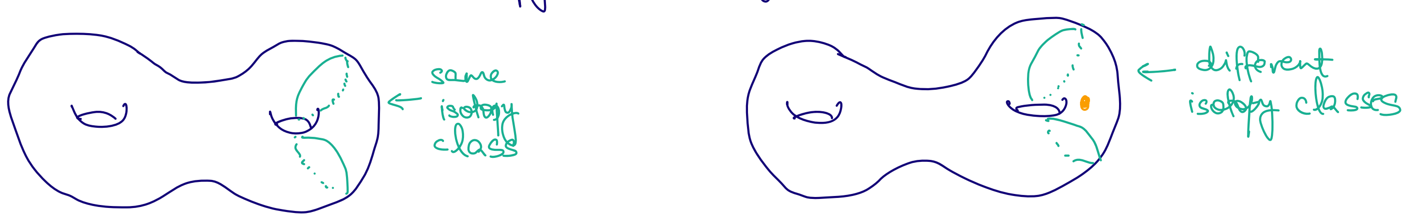
Harer's theorem says that $C(S_{g,1}) \simeq \vee S^{2g-2}$ and $C(S_{g,0}) \simeq \vee S^{2g-2}$

When $n=1$, we have a well-defined "forget a point" map $C(S_{g,1}) \xrightarrow{f} C(S_{g,0})$, but no bad curves.

$$\boxed{\text{Thm: } f: C(S_{g,1}) \to C(S_{g,0}) \text{ is a homotopy equivalence}}$$

Note that $\dim C(S_{g,1}) = 3g-3$ and $\dim C(S_{g,0}) = 3g-4$, though they have the same homotopy dim. of $2g-2$.

The extra dimension arises from the fact that the puncture creates an added isotopy class of curves.



Thus, forgetting the puncture corresponds to collapsing some simplices of $C(S_{g,1})$ down by a dimension.

The above theorem - whose proof is in Section IV — says that this collapsing of simplices does not change the homotopy type of $C(S_{g,1})$.

Assuming that $C(S_{g,1}) \simeq C(S_{g,0})$, we will prove that:

1. $C(S_{g,1})$ is $(2g-3)$-connected
2. $C(S_{g,0})$ has vanishing H_* in deg $\geq 2g-1$

This will prove that $C(S_{g,1}) \simeq \bigvee S^{2g-2} \simeq C(S_{g,0})$

III.1 : $C(S_{g,1})$ is $(2g-3)$-connected

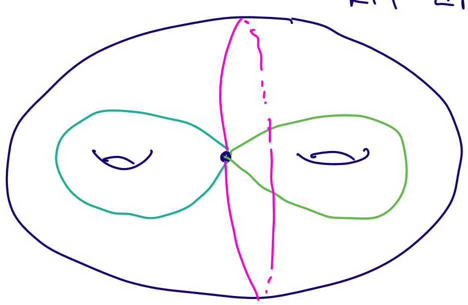
We will show that $C(S_{g,1})$ is $\simeq \mathcal{A}_\infty$, where $\mathcal{A}_\infty$ is "the arc complex at ∞", and show $\mathcal{A}_\infty$ is $(2g-3)$-connected.

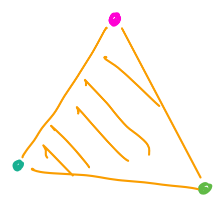
Step 1: $\mathcal{A}_\infty$ is $(2g-3)$ connected

Defn: The Arc Complex $\mathcal{A}$ $(n \geq 1)$

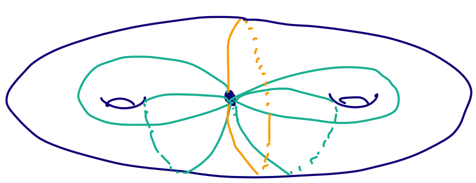
k-simplices $\longleftrightarrow$ 'arc systems'

k+1 arcs disjoint except maybe at endpts

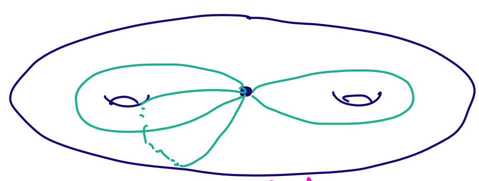




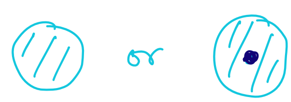
Defn: The arc complex at infinity $\mathcal{A}_\infty$



filling



non-filling

A filling arc system: cuts $S_{g,n}$ into disks: 

$\mathcal{A}_\infty$: subcomplex of $\mathcal{A}$ spanned by non-filling arc systems

An Euler characteristic argument shows that we need $\geq 2g$ arcs in any filling arc system.

Thus, $\mathcal{A}^{(2g-2)} \subset \mathcal{A}_\infty$.

Thm: $\mathcal{A}$ is contractible

This can be shown via a Hatcher flow argument, as described below.

Since $\mathcal{A}_\infty$ contains the $(2g-2)$-skeleton of a contractible simplicial complex, it follows that $\boxed{\mathcal{A}_\infty \text{ is } (2g-3)\text{-connected}}$

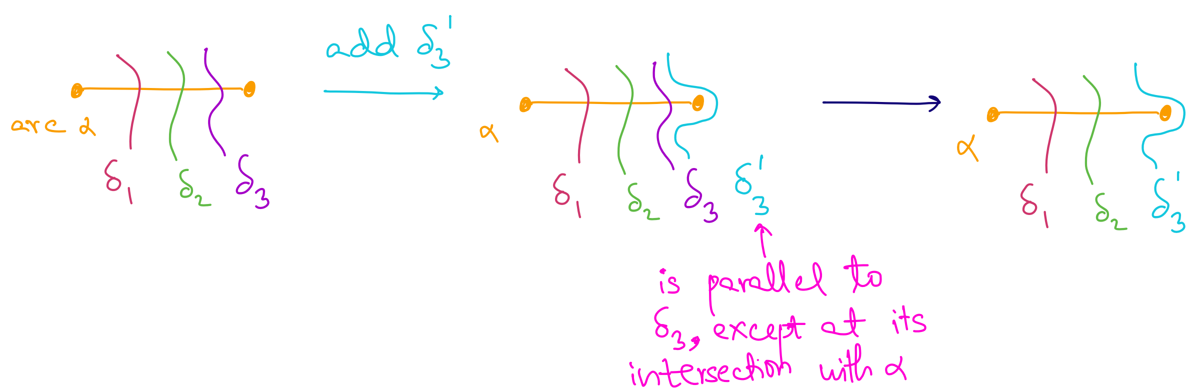
Hatcher flow on $\mathcal{A}$

Want to show that any (simplicial) map $\Psi: S^k \to \mathcal{A}$ is nullhomotopic.

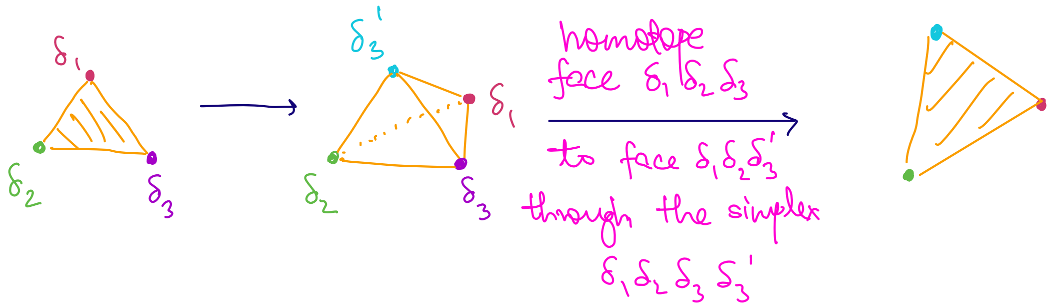
Fix an arc α. Will homotope Ψ so its img lies in lk α. We'll then be able to nullhomotope Ψ so its img is α.

The idea of Hatcher flow:

On $S_{g,1}$:



In $\mathcal{A}(S_{g,1})$:



Continue in this way until all arcs have been homotoped off of α.

Step 2: $C(S_{g,1}) \simeq \mathcal{A}_\infty$

We will work with the barycentric subdivisions of these complexes. Thus vertices correspond to curve (arc) systems and k-simplices correspond to flags of $(k+1)$ curve (arc) systems.

We will use the following Theorem due to Quillen, whose proof is in section V :

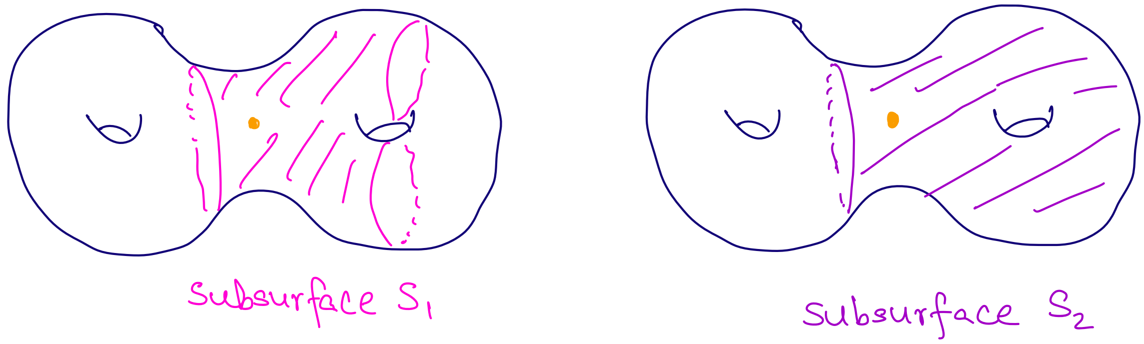
Quillen Fiber Lemma : A poset map $\phi : P \to Q$ is a homotopy equiv. if all fibers $\phi_{\leq q}$ $(= \{ p \in P : \phi(p) \leq q \})$ are contractible

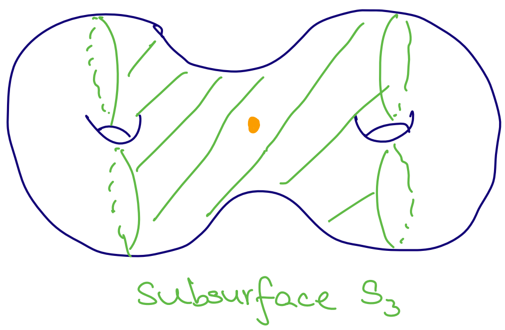
We will apply this Lemma twice.
We'll define the "subsurface complex" SC, and use the Lemma to construct htpy equiv. $C(S_{g,1}) \to SC$ and $\mathcal{A}_\infty \to SC$

The Subsurface Complex SC

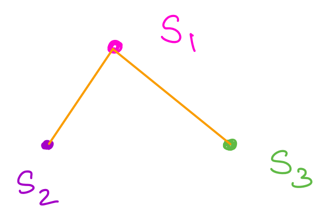
vertices $\longleftrightarrow$ connected subsurfaces of $C(S_{g,1})$ that
- contain the puncture
- bdry forms a curve system, possibly w/ 2 parallel copies of the same curve

k-simplices $\leftrightarrow$ chains of inclusions of subsurfaces



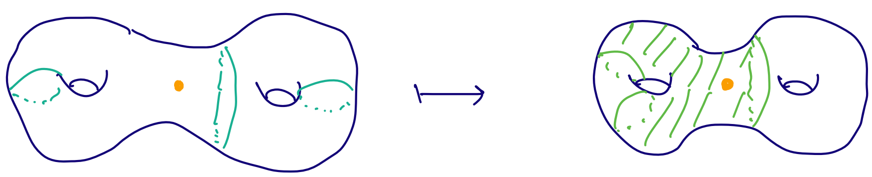


In SC :



$C(S_{g,1}) \simeq SC$

Given a curve system $C(S_{g,1})$, we can cut along the curves to separate $S_{g,1}$ into several subsurfaces. Mapping the curve system to the unique subsurface that contains the puncture defines a (poset) map $(\mathrm{sd}\, C(S_{g,1})) \xrightarrow{\phi} SC$



This is an order-reversing poset map. Each downward fiber $\phi_{\leq S}$ is a cone with cone point given by the curve system corresponding to ∂S.

Thus $C(S_{g,1}) \simeq SC$

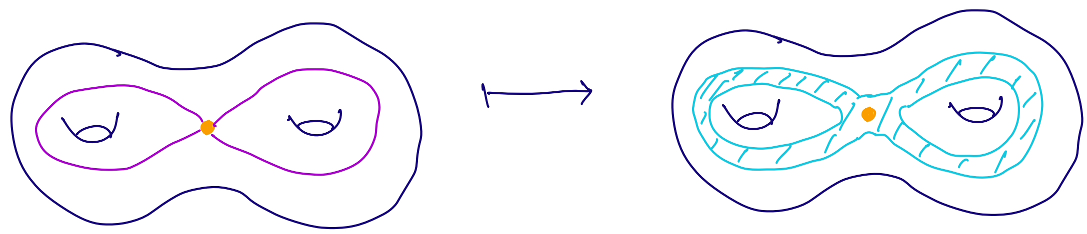
$\mathcal{A}_\infty \simeq SC$

Given an arc system, taking the union of annular neighbourhoods of each arc gives a subsurface.

Thus we get an order-preserving poset map $\psi : \mathcal{A}_\infty \to SC$

Each downward fiber $\psi_{\le S}$ consists of arc systems on the (punctured) surface S, and is thus $\cong \mathcal{A}(S)$, which is contractible.

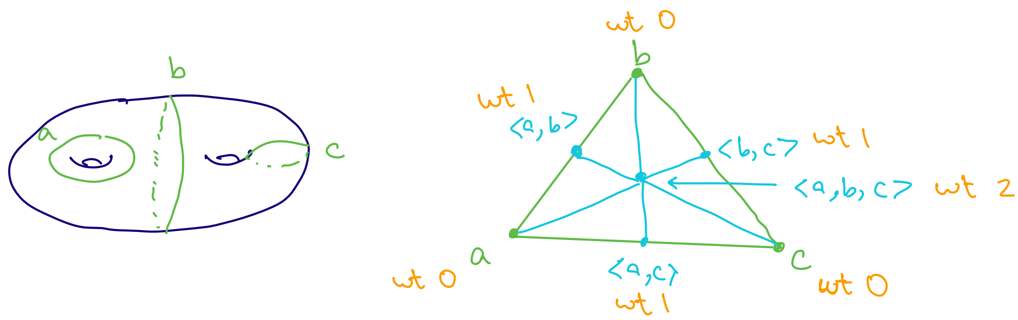
Thus $\mathcal{A}_\infty \simeq SC$



III.2 : $H_*(C(S_{g,0})) = 0$ for $* \ge 2g-1$

We shall work with the barycentric subdivision of $C(S_{g,0})$.

Let $X = sd(C(S_{g,0}))$. Thus vertices of X correspond to curve systems on $S_{g,0}$, and simplices correspond to flags of curve systems.



To each vertex of X, we can assign a weight. Vertices corresponding to a 1-curve system have wt 0, those for a 2-curve system have wt 2, and so on.

Let $X_k :=$ subcomplex of X spanned by the weight $\ge k$ vertices.

Thus $X_0 = X$, and X_{3g-4} is a discrete set of vertices.

Note that for a vertex γ in X_k of weight k, $\mathrm{lk}_{X_k}\gamma \subset X_{k+1}$.
Thus X_k is built out of X_{k+1} by coning off subcomplexes of X_{k+1}.

Here is a crucial Lemma for our argument:

Lemma: For a wt k vertex γ in X_k, its link is $\simeq \vee S^m$, where $m \le 2g-3$.

$$\boxed{\text{In particular, } H_*(\mathrm{lk}_{X_k}\gamma) = 0 \text{ for } * \ge 2g-2}$$

We defer the proof of this Lemma to the end of this section.
Assuming this Lemma, our next crucial claim is as follows:

Claim: Suppose γ is a wt k vertex, and let C be a representative of a simplicial ℓ-cycle in X_k. Thus $C = n_1\sigma_1 + \dots + n_r\sigma_r$ for ℓ-simplices $\sigma_1, \dots, \sigma_r$, with $\ell \ge 2g-1$.
Then we can replace C with a homologous chain so that none of the σ_i have γ as a vertex (and so that we don't increase the number of wt k vertices appearing on simplices in the chain.)

Proof: While reading the proof, it may help to also look at the pictorial examples that follow, that illustrate the proof idea.

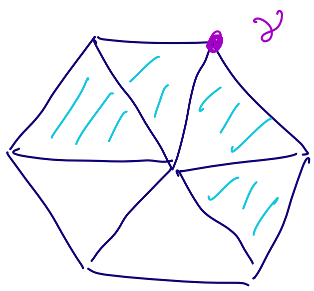
Let $C = n_1\sigma_1 + \dots + n\sigma_r$. Let S be the subchain of C spanned by simplices having γ as a vertex. Assume WLOG that $S = n_1\sigma_1 + \dots + n_t\sigma_t$. Note that $\partial S \subset \mathrm{Star}(\gamma)$. In fact, since C is a cycle, we can argue that ∂S does not intersect with γ, and thus $\partial S \subset \mathrm{lk}(\gamma)$. Also, since $\partial(\partial S) = 0$, ∂S is a $(\ell-1)$-cycle in $\mathrm{lk}\,\gamma$.

Since $\ell - 1 \ge (2g-1) - 1 = 2g-2$. Thus by the preceeding Lemma, $H_{\ell-1}(\mathrm{lk}(\gamma)) = 0$. Thus ∂S is the bdry of some ℓ-chain L in $\mathrm{lk}(\gamma)$.

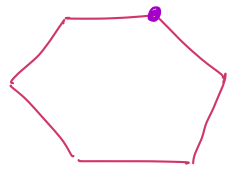
Now, consider the $(\ell+1)$-chain $\gamma * L$. It's bdry is $\gamma * \partial L - L = \gamma * \partial S - L = S - L$. Thus S is homologous to L.

Replacing C by $C - S + L$, the claim follows.

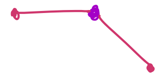
Example 1



X_k

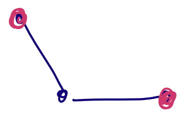


1-Cycle C

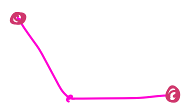


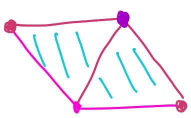
Subchain S

$\partial S \subset lk\,\gamma$ $\qquad$ ∂S is a 0-cycle in $lk\,\gamma$



$\tilde{H}_0(lk\,\gamma) = 0 \implies$ ∂S is the bdry of a 1-chain L in $lk\,\gamma$



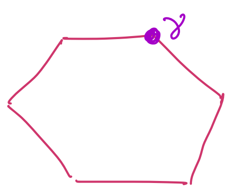


Take $\gamma * L$: Is a 2-chain with

$$\text{bdry} = (\gamma * \partial L) \cup L$$
$$= S \cup L$$

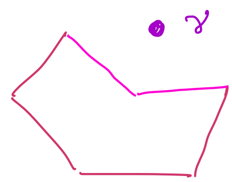
Thus S is homologous to L

Replace



C

with

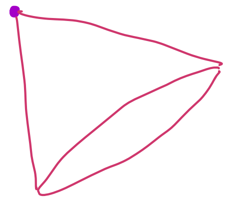


$C - S + L$

$C - S + L$ avoids γ.

Example 2

(1) γ

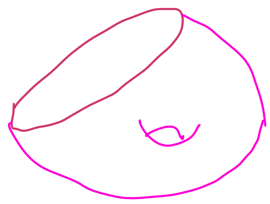


2-chain S

(2)

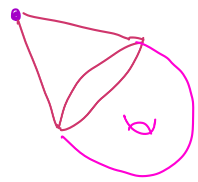
∂S is a 1-cycle in $\mathrm{lk}\,\gamma$

(3)



If $H_1(\mathrm{lk}\,\gamma) = 0$, then ∂S is ∂L for some 2-chain $L \subset \mathrm{lk}\,\gamma$

(4) Take $\gamma * L$:



It's bdry is $S - L$.

So now replace C with $C - S + L$. Both cycles are homologous, and the latter avoids γ.

— α — α — α — α — α — α — α — α —

Proof of the Lemma

Lemma: For a wt k vertex γ in X_k, its link is $\simeq \vee S^m$, where $m \le 2g-3$.

Suppose $\gamma = \langle \gamma_0, \dots, \gamma_k \rangle$, where the γ_i are the curves in the curve system γ. Cutting $S_{g,0}$ along these curves divides it into t surfaces $S_1, S_2, \dots, S_t$, with genus $g_1, \dots, g_t$ and no. of bdry components $r_1, \dots, r_t$.

Note that $\sum r_i = 2(k+1)$. Note that $\mathrm{lk}_{X_k}(\gamma) \cong C(S_1) * \dots * C(S_t)$

Now, by an Euler characteristic argument, we have $2g-2 = \sum_{i=1}^{t} (2g_i + r_i - 2)$
$= \sum_{i=1}^{t} (2g_i + r_i - 3) + t$. Now, each $C(S_i)$ is $\simeq \vee S^l$, with $l = 2g_i + r_i - 3$ if $g_i \ge 1$, and $l = 2g_i + r_i - 4$ if $g_i = 0$. Thus $\underset{1 \le i \le t}{*} C(S_i)$ is $\simeq \vee S^m$ with $m \le \sum(2g_i + r_i - 3) + (t-1)$

$= 2g - 2 - 1 = 2g-3$.

IV Proof of $C(S_{g,1}) \simeq C(S_{g,0})$

The forget-a-point map from $C(S_{g,1}) \to C(S_{g,0})$ is a homotopy equivalence.

This can be proved using Quillen's Fiber Lemma.

For a full proof, see Hatcher-Vogtman "Tethers & Hom. Stability for Surfaces", Section 4 Prop 4.7.

(V) Appendix

(I) The $A * X$ Lemma

Lemma: X: any simplicial complex, A: any discrete set
$Y \subset A * X$ and $A, X \subseteq Y$ s.t. $\forall a \in A$,
$lk_Y(a) \hookrightarrow X$ is a htpy equiv.
Then $Y \hookrightarrow A * X$ is a htpy equiv.

Proof: We want to show that any simplicial map $\varphi: S^k \to A * X$ can be homotoped to have its image $\subset Y$.
Since any such φ has only finitely many vertices of A in its image, it is enough to consider $|A|$ finite.
We'll induct on $|A|$.

Base Case: $|A| = 1$
Thus $Y \subset \{a\} * X$. Now $lk_Y(a) \xrightarrow{\simeq} X$ implies that X deformation retracts to $lk_Y(a)$.
We can use this retraction to deformation retract $\{a\} * X$ to Y. Thus $Y \xrightarrow{\simeq} \{a\} * X$

Induction Step: Let $A = \{a_1, \ldots, a_{n-1}, a_n\}$.
Let $A' = \{a_1, \ldots, a_{n-1}\}$, and Y' be the subcomplex of Y obtained by deleting a_n, and Y'' the one obtained by deleting $a_1, a_2, \ldots, a_{n-1}$. Thus $Y = Y' \cup Y''$.
We can assume that $Y' \xrightarrow{\simeq} A' * X$ and $Y'' \xrightarrow{\simeq} \{a_n\} * X$. We want $Y' \cup Y'' \hookrightarrow A * X$ to be a homotopy equiv.
We'll use the following Lemma, taken from Mather - "The Homology of a Lattice"

Lemma: Let X, Y, W, Z be simplicial complexes. Suppose $F: X \cup Y \to W \cup Z$ is s.t. $F|_X: X \to W$, $F|_Y: Y \to Z$, and $F|_{X \cap Y}: X \cap Y \to W \cap Z$ are htpy equiv. Then F is a htpy equiv.

We prove this Lemma at the end of the proof.

To apply it to our case, we need only show that

$Y' \cap Y'' \hookrightarrow (A' * X) \cap (\{a_n\} * X)$ is a htpy equiv.

Note: $Y' \cap Y'' = (\mathrm{lk}(a_1) \cap \mathrm{lk}(a_n)) \cup \dots \cup (\mathrm{lk}(a_{n-1}) \cap \mathrm{lk}(a_n))$

$(A' * X) \cap (\{a_n\} * X) = X$

We can use the Mather Lemma combined with the following Lemma to show that $Y' \cap Y'' \hookrightarrow (A' * X) \cap (\{a_n\} * X)$ is a htpy equiv

Lemma: If $P \overset{\simeq}{\hookrightarrow} R$ and $Q \overset{\simeq}{\longrightarrow} R$, then $P \cap Q \overset{\simeq}{\hookrightarrow} R$

Pf: Use excision on π_k to argue that

$\pi_k(P, P \cap Q) \cong \pi_k(R, Q) \cong 0$ for all k.

Thus $P \cap Q \overset{\simeq}{\hookrightarrow} P \overset{\simeq}{\hookrightarrow} R$

Since $\mathrm{lk}(a_i) \overset{\simeq}{\hookrightarrow} X \;\forall i$, the above Lemma implies

$\mathrm{lk}(a_i) \cap \mathrm{lk}(a_n) \overset{\simeq}{\hookrightarrow} X$.

We can now repeatedly use the Mather Lemma and the above excision lemma to argue that $(\mathrm{lk}(a_1) \cap \mathrm{lk}(a_n)) \cup \dots \cup (\mathrm{lk}(a_{n-1}) \cap \mathrm{lk}(a_n)) \hookrightarrow X$ is a homotopy equiv.

Proof of the Mather Lemma

We will use the fact that $F: X \to Y$ is a htpy equiv. iff the mapping cylinder $M(F)$ deformation retracts to X.

In the setting of the Lemma, we have $F: X \cup Y \to W \cup Z$.

$F|_{X \cap Y}$ is a htpy equiv, so $M(F|_{X \cap Y})$ deformation retracts to $X \cap Y$. Thus we have $H: M(F|_{X \cap Y}) \times I \to M(F|_{X \cap Y})$ s.t. H restricted to $M(F|_{X \cap Y}) \times \{0\}$ is the identity on $M(F|_{X \cap Y})$.

Use the homotopy extension property of CW-complexes to extend H to $\bar{H}: M(F|_X) \times I \to M(F|_X)$. Note that $\bar{H} = \mathrm{id}$ on $M(F|_X) \times \{0\}$

Now extend $\bar{H}$ to $\hat{H}: M(F) \times I \to M(F)$. Note that :—

$\hat{H}: M(F) \times \{0\} \to M(F)$ is identity, $\hat{H}(M(F|_X) \times I) \subset M(F|_X)$, $\hat{H}(M(F|_Y) \times I) \subset M(F|_Y)$, $\hat{H}(z,t) = z \;\forall z \in X \cap Y$. We can thus compose $\hat{H}$ with the deformation retractions of $M(F|_X)$, $M(F|_Y)$ onto X, Y respectively to get the desired deformation retract $M(F)$ to $X \cup Y$.

(II) Quillen's Fiber Lemma

Thm: A poset map $\phi : P \to Q$ is a homotopy equiv. if all fibers $\phi_{\leq q}$ $(= \{p \in P : \phi(p) \leq q\})$ are contractible.

Proof: We'll use the $\phi_{\leq q} \simeq *$ condition to construct a homotopy inverse $g : \Delta(Q) \to \Delta(P)$ ($\Delta(P)$ is the simplicial complex associated to the poset P)

Step 1: Constructing g

We'll construct g skeleton by skeleton.

On Vertices: vertices $\leftrightarrow$ elts $q \in Q$.

$\phi_{\leq q} \simeq *$, hence is non-empty, so pick $g(q) \in \phi_{\leq q}$

On Edges: edges $\leftrightarrow$ $q_0 < q_1$ in Q

$\phi_{\leq q_0} \subset \phi_{\leq q_1} \simeq *$. Since $\phi_{\leq q_1}$ is 0-conn; we can join $g(q_0), g(q_1) \in \phi_{\leq q_1}$ with a path.

Map the edge $q_0 < q_1$ to this path.

On 2-simplices: 2-simplices $\leftrightarrow$ $q_0 < q_1 < q_2$

$\phi_{\leq q_2}$ is 1-conn., thus we can fill in the loop formed by $g(q_0 < q_1)$, $g(q_1 < q_2)$, $g(q_0 < q_2)$ with a disk.

Map $q_0 < q_1 < q_2$ to this disk

... and so on ...

Step 2 : Checking ϕ, g are homotopy inverses

We want to show $g \circ \phi \simeq id_{\Delta(P)}$ and $\phi \circ g \simeq id_{\Delta(Q)}$.
We'll construct a homotopy $g \circ \phi \simeq id_{\Delta(P)}$ skeleton-by-skeleton. The $\phi \circ g$ case will be similar.

On Vertices : Note that $p, g \circ \phi(p) \in \phi_{\leq \phi(p)} \simeq *$
Thus there is a path joining p and $g \circ \phi(p)$
Use this path to homotope $g \circ \phi(p)$ to p.

On Edges : Suppose $p_0 < p_1$ is an edge.
The paths $[p_0, p_1], [p_1, g \circ \phi(p_1)], [g \circ \phi(p_1), g \circ \phi(p_0)]$, $[g \circ \phi(p_0), p_0]$ bound a disk in $\phi_{\leq \phi(p_1)} \simeq *$.

Homotope $[p_0, p_1]$ to $[g \circ \phi(p_0), g \circ \phi(p_1)]$ using this disk.

. . . and so on . . .