Reference: For definitions of collapsible, nonevasive, shellable, constructible, vertex-decomposable, pure, strongly connected, Cohen-Macaulay, see "Bjorner - Combinatorial Topology" Pg 1853 - 1856.

Notation: Δ is a finite simplicial complex (unless specified otherwise), $\dim \Delta = d$.
$\sigma \in \Delta$ is a simplex, of dim k.

Basic Facts About Links and Collapses

Ex 1: If $\sigma \cup \tau \in \Delta$ and $\sigma \cap \tau = \phi$, prove that $lk_{lk_\Delta(\sigma)}(\tau) = lk_\Delta(\sigma \cup \tau)$

Ex 2: Link of a simplex in a pure Δ is pure, of dim $d-k-1$ ($k = \dim \sigma$, $d = \dim \Delta$)

Ex 3: Prove that after collapsing a k-face σ of an n-simplex τ ($k<n$), the resultant complex is pure $(n-1)$-dimensional.

Cone => Nonevasive => Collapsible

Ex 4: Prove cone => nonevasive (Use induction on no. of vertices in Δ)
(Hint: Pick a cone point p. Let $q \in \Delta^\circ \setminus \{p\}$. Use the inductive hypothesis on $dl_\Delta(q)$.)

Ex 5: Suppose Δ is a (possibly infinite) complex and $x \in \Delta^\circ$ has a finite collapsible link $lk_\Delta(x)$. The goal of this exercise is to show that Δ can be collapsed to $dl_\Delta(x)$ by a finite no. of collapses.

1. Show this is true if $lk_\Delta(x) = \{\bullet\}$
2. If $\sigma \subset \tau \in lk_\Delta(x)$ s.t. σ can be collapsed to τ in $lk_\Delta(x)$, show that $\sigma \cup \{x\}$ can be collapsed to $\tau \cup \{x\}$ in Δ.
3. Let Δ' be the complex resulting after step (2). Show that $lk_{\Delta'}(x)$ is collapsible, and note that it has fewer simplices than $lk_\Delta(x)$.
4. Prove via induction that Δ can be collapsed to $dl_\Delta(x)$.

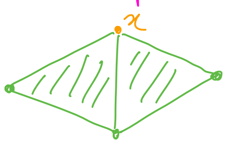
Ex 6: Assume Δ finite. Use Ex 5 to inductively show that Δ nonevasive $\Rightarrow$ Δ collapsible.

Nonevasiveness and Barycentric subdivision

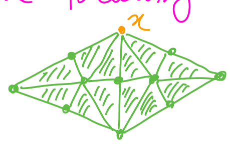
Ex 7: Alternative phrasing of non-evasiveness: Δ is nonevasive if we can successively delete vertices w/ nonevasive links to get a pt

Ex 8: $x \in \Delta^\circ$. Show that
$$\mathrm{lk}_{sd(\Delta)}(x) \underset{\text{simplicial iso}}{\cong} sd(\mathrm{lk}_\Delta(x))$$
(Can you also describe this isomorphism?)

Ex 9: Consider $x \in \Delta^\circ$. Take $\mathrm{dl}_{sd\,\Delta}(x)$. Think of vertices in this complex as corresponding to simplices $\{v_0, \dots, v_n\} \in \Delta$.
If it's helpful, first consider the following example:



Δ



$sd(\Delta)$

Prove that:

1. For any vertex $\{x, v\}$ in $\mathrm{dl}_{sd\Delta}(x)$, $\mathrm{lk}_{\mathrm{dl}_{sd(\Delta)}(x)}(\{x,v\})$ is a cone w/ cone pt. $\{v\}$, and thus nonevasive.
Also, for $v \neq w$, there is no edge b/w $\{x,v\}$ and $\{x,w\}$.
Successively delete these vertices, in the sense of Ex 4.

2. Consider the remaining complex (after deleting all $\{x,v\}$).
Show that all vertices of the form $\{x,v,w\}$ have links that are a cone w/ cone pt $\{v,w\}$.
We can successively delete these vertices.

3. Use the ideas from ① and ② to show that $\mathrm{dl}_{sd\Delta}(x)$ can be reduced to $sd(\mathrm{dl}_\Delta(x))$ by a sequence of the moves described in Ex 5.
Thus,
if $sd(\mathrm{dl}_\Delta(x))$ is nonevasive, then $\mathrm{dl}_{sd\Delta}(x)$ is nonevasive.

Ex 10: Use Ex 7, 8 & 9 to prove (for finite Δ) that non-evasiveness is preserved under taking barycentric subdivisions.

Collapsibility and Barycentric subdivision

Ex 11 : Suppose $\sigma = [v_0, v_1, \dots, v_k]$ and $\tau = [v_0, v_1, \dots, v_k, \dots, v_n]$

Let collapse(σ, τ) be the resultant complex after collapsing σ onto τ.

Show that collapse(σ,τ) can be obtained after a sequence of elementary collapses. (i.e. where the collapsing face is of codim 1).

Hint: We need to eventually remove all simplices containing $[v_0, v_1, \dots, v_k]$ as a face.

Step 1: Show that we can perform an elementary collapse to remove all $(n-1)$-simplices γ s.t. $[v_0, v_1, \dots, v_k] \subset \gamma \subset [v_0, v_1, \dots, v_{n-1}]$ and n-simplices π s.t. $[v_0, v_1, \dots, v_k, v_n] \subset \pi \subset [v_0, v_1, \dots, v_n]$

Step 2: Show that we can perform elementary collapses to remove all $(n-2)$-simplices γ s.t. $[v_0, v_1, \dots, v_k] \subset \gamma \subset [v_0, v_1, \dots, v_{n-1}]$ and $(n-1)$-simplices π s.t. $[v_0, v_1, \dots, v_k, v_n] \subset \pi \subset [v_0, v_1, \dots, v_n]$

Step 3: Guess (and prove!) what step 3 should be.

Ex.12 : Let $\sigma = [v_0, v_1, \dots, v_{n-1}]$ and $\tau = [v_0, v_1, \dots, v_k, \dots, v_n]$

The aim of this exercise is to prove that we can perform a sequence of collapses to reduce $sd(\tau)$ to $sd(\text{collapse}(\sigma,\tau))$.

We shall do this via elementary collapses.

Note that :

- vertices in $sd\,\tau$ correspond to faces of τ, and simplices correspond to flags.
- we need to get rid of all simplices that contain the vertices $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$
- It may be helpful to draw pictures for some concrete small values of n while doing this exercise.

(1) We'll first remove all n-simplices containing $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$

Let $\pi \in S_{[0,1,\dots,n]}$

(1.1) Suppose $\pi^{-1}(n) = n$. Show that we can perform the following collapse:

$$\{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-1)}\}\}$$

$$\subset \{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-1)}, v_{\pi(n)}\}\}$$

(1.2) Having done the collapses in (1.1), now suppose $\pi^{-1}(n) = n-1$.

Show that we can perform the following collapse :

$$\{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_n, v_{\pi(n-1)}\}\}$$

$$\subset \{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_n\},$$
$$\{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_n, v_{\pi(n-1)}\}\}$$

(1.3) Having done the collapses in (1.2), now suppose $\pi^{-1}(n) = n-2$.

Show that we can perform the following collapse:

$$\{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}, v_n, v_{\pi(n-1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}, v_n, v_{\pi(n-1)}, v_{\pi(n)}\}\}$$

$$\subset \{\{v_{\pi(0)}\}, \{v_{\pi(0)}, v_{\pi(1)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}, v_n\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}, v_n, v_{\pi(n-1)}\},$$
$$\{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-3)}, v_n, v_{\pi(n-1)}, v_{\pi(n-2)}\}\}$$

(1.4) We can keep proceeding in this way. The $(n+1)$-th step is: $\pi^{-1}(n+1) = 0$. Collapse:

$$\{\{v_n, v_{\pi(1)}\}, \{v_n, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_n, v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n-1)}, v_{\pi(n)}\}\}$$

$$\subset \{\{v_n\}, \{v_n, v_{\pi(1)}\}, \{v_n, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_n, v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n-1)}, v_{\pi(n)}\}\}$$

2 Check that 1 successfully removes all n-simplices containing $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$.
Now we wanna remove $(n-1)$-simplices.
As before, let $\pi \in S_{[0,1,\dots,n]}$

(2.1) Suppose $\pi^{-1}(n) = n$. Show that we can perform the following collapse:

$$\{\{v_{\pi(0)}, v_{\pi(1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_{\pi(n-1)}\}\}$$

$$\subset \{\{v_{\pi(0)}, v_{\pi(1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_{\pi(n-1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-1)}, v_n\}\}$$

(2.2) Now suppose $\pi^{-1}(n) = n-1$. Show that we can perform the following collapse:

$$\{\{v_{\pi(0)}, v_{\pi(1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_{\pi(0)}, \dots, v_{\pi(n-2)}\}, \{v_{\pi(0)}, \dots, v_{\pi(n-2)}, v_n, v_{\pi(n)}\}\}$$

$$\subset \{\{v_{\pi(0)}, v_{\pi(1)}\}, \{v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}\}, \dots, \{v_{\pi(0)}, \dots, v_{\pi(n-2)}\}, \{v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(n-2)}, v_n\},$$
$$\{v_{\pi(0)}, \dots, v_{\pi(n-2)}, v_n, v_{\pi(n)}\}\}$$

(2.3) Keep proceeding in this way. The $(n+1)$th step is $\pi^{-1}(n) = 0$ and:

$\{\{v_n, v_{\pi(1)}, v_{\pi(2)}\}, \{v_n, v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}\}, \dots, \{v_n, v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)}\}\}$

$\subset \{\{v_n, v_{\pi(1)}\}, \{v_n, v_{\pi(1)}, v_{\pi(2)}\}, \{v_n, v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}\}, \dots, \{v_n, v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(n)}\}\}$

3. Check that 2 successfully removes all $(n-1)$-simplices containing $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$.
   Formulate the strategy for now removing all $(n-2)$-simplices containing $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$
4. We can keep doing this, until we have removed all simplices containing $\{v_0, v_1, \dots, v_{n-1}\}$ or $\{v_0, v_1, \dots, v_{n-1}, v_n\}$.
   What's left is $sd(\text{collapse}(\sigma, \tau))$.

Ex 13: Use Ex 11 & 12 to show that collapsibility is preserved under taking barycentric subdivisions.

Vertex-Decomposable $\Rightarrow$ Shellable $\Rightarrow$ Constructible

Ex 14: For a finite pure d-dim Δ, prove the following definitions of shellability are equivalent:

1. Δ can be reduced to a d-simplex by a sequence of (k,d)-collapses, $k \le d$.
2. The d-faces of Δ can be ordered $\sigma_1, \sigma_2, \dots, \sigma_n$ so that $\forall k$,
   $$(\partial\sigma_1 \cup \partial\sigma_2 \cup \dots \cup \partial\sigma_{k-1}) \cap \partial\sigma_k$$
   is a pure $(d-1)$-dim complex.
3. The d-faces of Δ can be ordered $\sigma_1, \sigma_2, \dots, \sigma_n$ so that $\forall i < k$, $\exists j < k$ s.t. $\sigma_i \cap \sigma_k \subset \sigma_j \cap \sigma_k$ and $\sigma_j \cap \sigma_k$ is $\dim(d-1)$.

Ex 15: Compare and contrast the definitions of:
(1) shellable and collapsible
(2) vertex-decomposable and nonevasive

Ex 16: Mimic Ex 6 to show that Δ vertex decomposable $\Rightarrow$ Δ shellable.

Ex 17: Mimic Ex 10 & 13 to show that vertex-decomposability and shellability are preserved by barycentric subdivision.

Ex 18: Show that shellable $\Rightarrow$ constructible.

Shellability & Constructibility are preserved by links

Ex 19: Use Condition ③ of Ex 14 to show that if $\sigma \in \Delta$ (simplex) and Δ is shellable, then $lk_\Delta(\sigma)$ is shellable.

Ex 20: The goal of this exercise is to prove that constructibility is inherited by links. Let $d = \dim \Delta$, $\sigma \in \Delta$ be of dim k.

① If Δ is a single simplex, show that $lk_\Delta(\sigma)$ is a simplex (and thus constructible).

② Show that if $\Delta = \Delta_1 \cup \Delta_2$ and $\sigma \in \Delta_1 \setminus \Delta_2$, then $lk_\Delta(\sigma) \subset \Delta_1$.

③ Show that if $\sigma \in \Delta_1 \cap \Delta_2$, then $lk_\Delta(\sigma) = lk_{\Delta_1}(\sigma) \cup lk_{\Delta_2}(\sigma)$ and $lk_{\Delta_1 \cap \Delta_2}(\sigma) = lk_{\Delta_1}(\sigma) \cap lk_{\Delta_2}(\sigma)$.

④ Show that if $d = 0$ or 1, then $lk_\Delta(\sigma)$ is always constructible.

⑤ For $d \geq 2$, show by inducting on the dim d and also on the size of the vertex set of Δ that Δ constructible $\Rightarrow lk_\Delta(\sigma)$ is constructible.

Properties of Cohen-Macaulay Complexes

Ex 21: Use Ex 20 to prove that shellable $\Rightarrow$ constructible $\Rightarrow$ homotopy CM

Ex 22: ① Suppose Δ is (homology) Cohen-Macaulay, and that $x \in \Delta^\circ$. Show that $H_i(\Delta, \Delta - \{x\}) = 0 \quad \forall \; i \neq d-1$.

② Now suppose that x is in the interior of a simplex $\sigma \in \Delta$ of dim k. Show in this case too that $H_i(\Delta, \Delta - \{x\}) = 0 \; \forall \, i \neq d-1$.

③ Prove that Δ is homology Cohen-Macaulay $\Leftrightarrow H_i(\Delta, \Delta - \{x\}) = 0$ for $i \neq d-1 \quad \forall \; x \in \|\Delta\|$.

Ex 23: Define property P as follows:

(P) : For all $\sigma \in \Delta \cup \{\emptyset\}$ s.t. σ is of codim ≤ 2 (i.e. $\dim(lk_\Delta(\sigma)) \geq 1$), $lk_\Delta(\sigma)$ is connected.

Suppose Δ is finite-dimensional (of dim d) and satisfies P. Our goal is to prove that Δ is pure and strongly connected. We will induct on dim Δ.

① Prove that the claim holds when $d=0$ or 1.

② Using the fact that $lk_{lk_\Delta(\sigma)}(\tau) = lk_\Delta(\sigma\cup\tau)$, show that if Δ satisfies property P, then so does $lk_\Delta(\sigma)$.

③ Suppose $d\geq 2$, and Δ is not pure. Show that we can find $x\in\Delta^0$ s.t. x belongs to 2 distinct maximal faces of Δ with different dimensions.
Thus conclude that $lk_\Delta(x)$ is not pure.
(Hint: • Let Δ' be the (complex formed by the) union of all dim d simplices in Δ.
• Since Δ is not pure, there must be a dim ≥ 1 simplex in $\Delta\setminus\Delta'$ (which must also necessarily have dim $< d$.
• $lk_\Delta(\phi)=\Delta$ is connected, so this implies that there must be some vertex $x\in\Delta'$ that is part of such a simplex.
Show that this is the desired point x.)

④ Suppose $d\geq 2$. Using ②, ③ and inducting on dim Δ, show that if Δ satisfies P, then Δ is pure.

⑤ Let $k=\dim\sigma$. Recall that (if Δ is pure) $\dim(lk_\Delta\sigma) = d-k-1$.
Show that $lk_{\Delta^{d-1}}(\sigma) = (lk_\Delta\sigma)^{d-k-2}$.

⑥ Show that if $d\geq 2$ and Δ is connected then Δ^{d-1} is also connected.
(Bonus: in fact, Δ strongly connected also implies that Δ^{d-1} is strongly connected)

⑦ Suppose $d\geq 2$. Show that Δ^{d-1} strongly connected $\Rightarrow$ Δ is strongly connected.
Hint: say that a pair of d-simplices (σ,τ) in Δ is strongly connected if we have a sequence of facets $\sigma=\sigma_1,\sigma_2,\dots,\sigma_n=\tau$ s.t. $\sigma_i\cap\sigma_{i+1}$ is d-dim $\forall i$.
Thus Δ is strongly-connected iff every pair is strongly connected.

① First show that, if Δ^{d-1} is strongly connected and $\sigma\cap\tau\neq\phi$, then (σ,τ) is strongly connected.

② Now suppose $\sigma\cap\tau=\phi$. Let σ',τ' be a $(d-1)$-dim face of σ,τ, resp.
Then we have a sequence $\sigma'=\sigma_1',\sigma_2',\dots,\sigma_n'=\tau'$ strongly-connecting (σ',τ') in Δ^{d-1}.

③ for $i=1,2,\dots,n$, pick a d-simplex $\sigma_i\in\Delta$ with σ_i' as a face, with $\sigma_1=\sigma$, $\sigma_n=\tau$.

④ Show that $\sigma_i\cap\sigma_{i+1}\neq\phi$ $\forall i$. Thus each pair (σ_i,σ_{i+1}) is strongly-connected.
Hence show that (σ,τ) is strongly-connected.

⑧ Suppose $d\geq 2$ and Δ satisfies property P. Use ⑤ and ⑥ to conclude Δ^{d-1} also satisfies property P.
Inductively assume Δ^{d-1} is strongly connected, and now use ⑦ to conclude that Δ is strongly connected.

Ex 24: Conclude from Ex 23 that if Δ is (homotopy or homology) CM, then Δ is pure and strongly connected.