* We are working only with discrete groups in this note.

(I) Coefficients in a $\pi_1 X$-module

(1) Defn: Suppose X top. space, $\tilde{X} \to X$ univ cover, G an abelian gp that is a $\pi_1 X$-module.
Note that $\pi_1 X \curvearrowright C_n(\tilde{X})$.

Can form (co)chain complexes from $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} G$ and $\mathrm{Hom}_{\mathbb{Z}[\pi_1 X]}(C_n(\tilde{X}), G)$.

(* We consider $\pi_1 X \curvearrowright C_n(\tilde{X})$ on the left. To form the tensor product, we convert this left action to a right action via $\sigma \cdot \gamma := \gamma^{-1}\sigma$ $\quad \gamma \in \pi_1 X$, $\sigma \in C_n(\tilde{X})$)

Then
$$H_*(X;G) := \text{homology of } (C_*(\tilde{X}) \otimes_{\pi_1 X} G)$$
$$H^*(X;G) := \text{cohomology of } (\mathrm{Hom}_{\pi_1 X}(C_*(\tilde{X}), G))$$

(2) Examples:

2.1 Gp cohomology is a special case of the above defn - take the gp to be $\pi_1 X$ and X to be its classifying space.

2.2 suppose $\pi_1 X \curvearrowright G$ trivially. Then, $C_n(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1 X]} G$
$$\cong \frac{C_n(\tilde{X})}{\mathbb{Z}\pi_1 X} \otimes_{\mathbb{Z}} G \cong C_n(X) \otimes_{\mathbb{Z}} G.$$

So the above agrees with the defn of homology of X with coefficients in an abelian group G.
Similarly, $\mathrm{Hom}_{\mathbb{Z}[\pi_1 X]}(C_n(\tilde{X}), G) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\frac{C_n(\tilde{X})}{\mathbb{Z}[\pi_1 X]}, G\right)$
so a similar statement holds for cohomology.

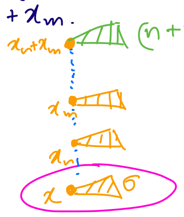
(II) Coefficients in a bundle of groups (over X)

(1) Defn: X a top. space, G an abelian gp, $E \to X$ a bundle of gps w/ fibre G.

(Homology) — We define $H_*(X;E)$ as follows :-
The chain groups $C_n(X;E)$ consist of finite sums $\sum n_i \sigma_i$, where $\sigma_i : \Delta^n \to X$ is an n-simplex of X and n_i is a lift of σ_i to E.

We add two lifts of a fixed σ, $n\sigma + m\sigma$, by picking a point x in $\mathrm{im}\,\sigma$, considering its images x_n, x_m under the lifts, and lifting σ s.t. x gets lifted to $x_n + x_m$.

Equivalently, add the two simplices pointwise: $(n+m)(s) = n(s) + m(s)$ $\forall s \in \Delta^n$



(not hard to check $(n+m)\sigma$ is independent of choice of x — show that simplices induce gp homos (in fact isos) b/w fibres, just like we did for paths in the Bundle of Gps note)

<u>Boundary maps</u>: $\partial(n\sigma) = \sum_{j=0}^{n} (-1)^j n_{|\sigma|[v_0,\dots,\hat{v}_j,\dots,v_n]}\, \sigma_{|[v_0,\dots,\hat{v}_j,\dots,v_n]}$

<u>Check</u>: $\partial^2 = 0$

Thus $H_*(X;E) := H_*(C_*(X;E), \partial)$

<u>Quick Example</u>: If $E \to X$ is the trivial product bundle $X \times G$, then the above defn recovers $H_*(X;G)$ for an abelian group G.

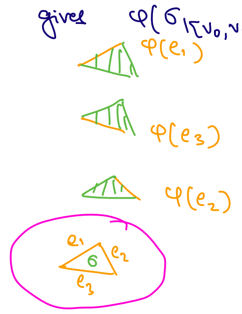
(Cohomology) — To define $H^*(X;E)$, we build a cochain complex as follows:

$C^n(X;E)$ consists of maps φ that assign to a simplex $\sigma : \Delta^n \to X$ a lift of σ $\varphi(\sigma) : \Delta^n \to E$.

<u>Coboundary maps</u> δ: $(\delta\varphi)(\sigma) = \sum (-1)^j \varphi(\sigma)_{[v_0,v_1,\dots,\hat{v}_j,\dots,v_{n+1}]}$

$\varphi \in C^n(X;E)$ $\quad \sigma : \Delta^{n+1} \to X$

where $\varphi(\sigma)_{[v_0,v_1,\dots,\hat{v}_j,\dots,v_{n+1}]}$ refers to the lift of σ whose restriction to the face $[v_0,v_1,\dots,\hat{v}_j,\dots,v_{n+1}]$ gives $\varphi(\sigma_{|[v_0,v_1,\dots,\hat{v}_j,\dots,v_{n+1}]})$



$(\delta\varphi)(\sigma)$ is the sum (upto $\pm$ signs) of the three simplices in green

<u>Quick Example</u>: As before, if $E \cong X \times G$, then this defn recovers the usual defn of $H^n(X;G)$ for G abelian.

② <u>Equivalence with coefficients in a $\pi_1 X$-module:</u>

For a given $\mathbb{Z}[\pi_1 X]$-module G, there is a unique (upto iso) bundle of gps $E \to X$ over X w/ fibre G that induces this given action of $\pi_1 X$ on G.
We shall show that (co)homology w/ coeffs in G is the same as that w/ coeffs in the bundle E.
Note further that E can be constructed as the quotient of $\tilde{X} \times G$ by $\pi_1 X$ acting diagonally. Thus an n-simplex σ of E corresponds to the orbit of a pair $(\tilde{\sigma}, g)$ in $\tilde{X} \times G$ under $\pi_1 X$.

(1) <u>Homology</u>: Identify $C_n(X;G)$ with $C_n(X;E)$ as follows :—
Take $m\sigma \in C_n(X;E)$. Thus $\sigma : \Delta^n \to X$ is an n-simplex and m is a lift of σ to E. Then m corresponds to the orbit of a pair

$(\tilde{\sigma}, g) \in \tilde{X} \times G$, where $\tilde{\sigma}$ is a lift of σ.

We can send $m\sigma \mapsto \tilde{\sigma} \otimes g$

This is well-defined since $\gamma\tilde{\sigma} \otimes \gamma g =: \tilde{\sigma}\gamma^{-1} \otimes \gamma g = \tilde{\sigma} \otimes \gamma^{-1}\gamma g$
$= \tilde{\sigma} \otimes g$

(2) Cohomology: Identify $C^n(X; E)$ with $C^n(X; G)$ as follows:

Given $\varphi \in C^n(X; E)$, φ assigns to each n-simplex σ of X a lift in E, which is equivalent to assigning σ the orbit of some $(\tilde{\sigma}, g) \in \tilde{X} \times G$ where $\tilde{\sigma}$ is a lift of σ.

The image of φ in $C^n(X; G)$ is a function that maps $\tilde{\sigma} \mapsto g$. Note that since $(\gamma\tilde{\sigma}, \gamma g)$ is in the same orbit, this means $\gamma\tilde{\sigma} \mapsto \gamma g$, so this is an elt of $\mathrm{Hom}_{\mathbb{Z}\pi_1 X}(C_n(\tilde{X}), G)$.

It is well-defined because each n-simplex of $\tilde{X}$ is a lift of an n-simplex of X (for eg it is a lift of its projected image in X)

III Properties

① For (co)homology w/ coeffs in a bundle, we can adapt many of the defns & results for ordinary singular (co)homology and prove:

- Induced maps f^*, f_* on (co)hom. by bundle maps f
- Homotopy invariance of induced maps
- (Co)homology of a pair (X, A); hence LES of a pair
- Excision
- Simplicial & cellular (co)homology w/ bundle coeffs
- Equiv. of simplicial /cellular with singular

For many of these properties, the key idea that carries over is that we can perform constructions like $\underbrace{\text{barycentric subdivision}}_{\text{eg: to prove excision}}$, $\underbrace{\text{prism operator}}_{\text{eg: to prove homotopy equivalence}}$, etc. on simplices on X as before

② Coeffs in $\mathbb{Z}[\pi]$ $(\pi = \pi_1 X)$

- $H_n(X; \mathbb{Z}[\pi]) \cong H_n(\tilde{X}; \mathbb{Z})$

  More generally, let $\pi' \subset \pi$ be a subgp, and $X' \to X$ the cover corresponding to it.

  Then $H_n(X; \mathbb{Z}[\pi/\pi']) \cong H_n(X'; \mathbb{Z})$

  Even more generally, for an abelian gp A, $H_n(X; A[\pi/\pi']) \cong H_n(X'; A)$

- For X a finite CW-complex,

$$H^n(X; \mathbb{Z}[\pi]) \cong H^n_c(\tilde{X}; \mathbb{Z})$$

(* note that the right hand side refers to compactly supported cohomology)

IV Duality for non-oriented manifolds (twisting of the coefficients)

Poincaré duality states that for a closed oriented n-manifold M, we have

$$H^k(M; \mathbb{Z}) \cong H_{n-k}(M; \mathbb{Z})$$

(in fact, we can take any coefficient ring R). For non-compact manifolds a similar result holds, but involving compactly supported cohomology.

Turns out we can prove a duality result for non-oriented closed manifolds — the loss of orientability coming at the cost of a "twisting of the coefficients".

Here's a sketch of how this works:

- We work w/ coeffs in a bundle of rings E. In this setting we can define cup & cap product analogously to the usual definitions.
- The Poincare duality map worked via taking cap product with a fundamental class.
  So we first show $H_n(M; E)$ has a fundamental class μ, in the sense that the image of μ under $H_n(M; E) \to H_n(M, M - \{x\}; E)$ is a ring generator of $H_n(M, M - \{x\}; E) \underset{\text{excision}}{\cong} R$, where R is the fibre ring of E.
- With this setting, we can now adapt the proof of Poincare duality and show that cap w/ μ gives an iso

$$H^k(M; E) \cong H_{n-k}(M; E)$$

Here's how we execute this strategy:

- For our duality result, we take our fibre ring to be $\mathbb{Z}[i]$.
  Recall that for non-oriented M, we had two $\pi_1(M)$-module structures on $\mathbb{Z}$. The trivial action, denoted $\mathbb{Z}$, and the orientation action, denoted $\tilde{\mathbb{Z}}$, where $\gamma \in \pi_1(M)$ acts via multiplication by ± 1 depending on whether γ is orientation preserving or not.

  We use these two actions to give a $\pi_1(M)$-action on $\mathbb{Z}[i]$. As a $\pi_1(M)$-module, $\mathbb{Z}[i] \cong \underbrace{\mathbb{Z}}_{\text{real part}} \oplus \underbrace{\tilde{\mathbb{Z}}}_{\text{imaginary part}}$. So if γ is orientation-preserving, then $\gamma(a+ib) = a+ib$.

  If not, then $\gamma(a+ib) = a-ib$.

  Note that $H_k(M; \mathbb{Z}[i]) \cong H_k(M; \mathbb{Z}) \oplus H_k(M; \tilde{\mathbb{Z}})$, and similarly for cohomology.

  The fund. class we construct lies in $H_n(M; \tilde{\mathbb{Z}})$. Because multiplication w/

i swaps real and imaginary parts, the duality iso

$$H^k(M; \mathbb{Z}[i]) \cong H_{n-k}(M; \mathbb{Z}[i])$$

actually splits into separate isos:

$$H^k(M; \mathbb{Z}) \cong H_{n-k}(M; \tilde{\mathbb{Z}}) \quad , \quad H^k(M; \tilde{\mathbb{Z}}) \cong H_{n-k}(M; \mathbb{Z})$$

This is the twisting of coefficients alluded to earlier.

So what's left is to construct this fundamental class.

<u>Outline of steps:</u> ① Show that we have an LES:

$$\cdots \to H_{n+1}(M; \mathbb{Z}) \to H_n(M; \tilde{\mathbb{Z}}) \to H_n(\tilde{M}; \mathbb{Z}) \xrightarrow{p_*} H_n(M; \mathbb{Z}) \to \cdots$$

($\tilde{M}$: 2-sheeted oriented cover of M)

② Since $\dim M = n$, we have $H_{n+1}(M; \mathbb{Z}) \cong 0$

And non-orientability $\Rightarrow H_n(M; \mathbb{Z}) \cong 0$

So this gives an iso $H_n(M; \tilde{\mathbb{Z}}) \cong H_n(\tilde{M}; \mathbb{Z}) \cong \mathbb{Z}$

A generator of $H_n(M; \tilde{\mathbb{Z}})$ then gives us our "imaginary fundamental class" — Note that it generates $H_n(M; \mathbb{Z}[i]) \cong H_n(M; \mathbb{Z}) \oplus H_n(M; \tilde{\mathbb{Z}})$ as a ring, just as $i \in \mathbb{Z}[i]$ generates $i\mathbb{Z}$ as a group and $\mathbb{Z}[i]$ as a ring.

③ To show it's a fund. class, we need to show that $\mathbb{Z} \cong H_n(M; \tilde{\mathbb{Z}}) \to H_n(M, M - \{x\}; \tilde{\mathbb{Z}}) \cong \mathbb{Z}$ is an iso $\forall x \in M$.

Here's an outline of this:

$$\begin{tikzcd}
H_n(M; \tilde{\mathbb{Z}}) \arrow[r, "\cong"] \arrow[d, "\text{Want } \cong \text{ here}"'] \arrow[dr, phantom, "③ \circlearrowright"] & H_n(\tilde{M}; \mathbb{Z}) \arrow[d, "② \cong"] \\
H_n(M, M - \{x\}; \tilde{\mathbb{Z}}) \arrow[r, "① j^*", "\cong"'] & H_n(\tilde{M}, \tilde{M} - \{\tilde{x}\}; \mathbb{Z})
\end{tikzcd}$$

① construct j^* using the $H_n(M; \tilde{\mathbb{Z}}) \to H_n(\tilde{M}; \mathbb{Z})$ map and show it's an iso

② $H_n(\tilde{M}; \mathbb{Z}) \to H_n(\tilde{M}, \tilde{M} - \{\tilde{x}\}; \mathbb{Z})$ by orientability

③ Show the diagram commutes

So now what's left is to construct the LES.

Fix an orientation-reversing loop γ of $\pi_1(M)$. Let $C_n^{\pm}(\tilde{M}) := \{\sigma \in C_n(\tilde{M}) \mid \gamma\sigma = \pm\sigma\}$

<u>Note</u>: $C_n^{\pm}(\tilde{M})$ is spanned by a basis $\sigma \pm \gamma\sigma$, for $\sigma \in C_n(\tilde{M})$.

We have an SES $0 \to C_n^+(\tilde{M}) \hookrightarrow C_n(\tilde{M}) \xrightarrow{\Delta} C_n^-(\tilde{M}) \to 0$, $\Delta(\sigma) = \sigma - \gamma\sigma$.

Note also that $C_n(\tilde{M}) \to C_n(\tilde{M}) \otimes_{\pi_1(M)} \tilde{\mathbb{Z}}$ has $\ker = C_n^+(\tilde{M})$.

So, $C_n(\tilde{M}) \otimes_{\pi_1(M)} \tilde{\mathbb{Z}} \cong C_n(\tilde{M}) / C_n^+(\tilde{M}) = C_n^-(\tilde{M})$.

$\therefore\ C_n^-(\tilde{M}) \cong C_n(\tilde{M}) \otimes_{\pi_1(M)} \tilde{\mathbb{Z}}$

Now the LES follows from the SES

$$0 \to C_n^-(\tilde{M}) \hookrightarrow C_n(\tilde{M}) \xrightarrow{\Sigma} C_n^+(\tilde{M}) \to 0 \qquad \Sigma(\sigma) = \sigma + \gamma\sigma$$

and noting that the homology of $\{C_n^-(\tilde{M})\} = \{C_n(\tilde{M}) \otimes_{\pi_1 M} \tilde{\mathbb{Z}}\}$ is precisely $H_n(M; \tilde{\mathbb{Z}})$

<u>Thought Exercise</u>: Think of a geometric picture to see what this fundamental class "looks like".