# I K(G,1) Spaces

## 1) Defn:

A $K(G,1)$ space $X$ is a space with $\pi_1(X) \cong G$ and a contractible universal cover $\tilde{X} \to X$.

Eg: $S^1$ is a $K(\mathbb{Z},1)$, $S^1 \times S^1$ is a $K(\mathbb{Z}^2,1)$, $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}_2,1)$, The univ. cover of $S^1 \vee S^1$ is a $K(F_2,1)$.

## 2) Existence:

Given $G$ we can always construct a $K(G,1)$ that is a simplicial complex, as follows:

Note that its equivalent to constructing a contractible space w/ covering space action of $G$.

Let $EG$ be a $\Delta$-complex, w/ $n$-simplices given by ordered $(n+1)$-tuples $[g_0, g_1, \dots, g_n]$, $g_i \in G$.
This $n$-simplex attaches to its faces $[g_0, g_1, \dots, \hat{g}_i, \dots, g_n]$ in the natural way.

EG is contractible: We have a homotopy $h_t$ that slides a point $x \in [g_0, g_1, \dots, g_n]$ to the vertex $[e]$ along the line segment joining $x$ to $e$ in the $(n+1)$-simplex $[e, g_0, g_1, \dots, g_n]$

Rmk: $h_t$ takes $e$ along the loop $[e,e]$, so it is not a deformation retraction.

$G \curvearrowright EG$ simplicially via $g \curvearrowright [g_0, g_1, \dots, g_n] = [gg_0, gg_1, \dots, gg_n]$

Rmk: Writing down the simplicial groups and boundary maps for EG gives the standard free resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules

## 3) Uniqueness upto Homotopy Type:

$K(G,1)$'s are unique upto homotopy, as implied by the following proposition.

Propn: $X$: connected CW-complex
$Y$: a $K(G,1)$
$\langle X, Y \rangle$: basept-preserving htpy classes of maps $(X,x_0) \to (Y,y_0)$

Then the map

$$\langle X, Y \rangle \longrightarrow \text{gp homs } \pi_1(X,x_0) \to \pi_1(Y,y_0)$$
$$f \longmapsto f_*$$

is a bijection

Proof: • Surjectivity: Given a gp. hom $\varphi: \pi_1(X,x_0) \to \pi_1(Y,y_0)$, want to construct $f: (X,x_0) \to (Y,y_0)$.

Will inductively construct $f$ on each $n$-skeleton.

0-skeleton: Send all vertices to $y_0$.

**1-skeleton:** Pick a maximal tree $T$ of $X'$ and send all of $T$ to $y_0$.

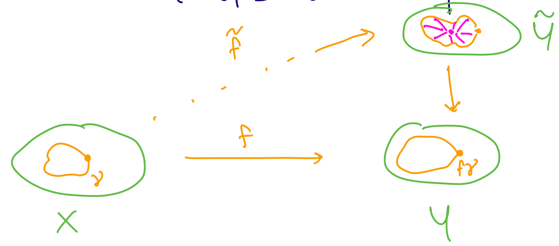
What's left are edges that give loops in $X'/T$, that give generators for $\pi_1(X,x_0)$.

Send these edge to loops in $Y$ centred at $y_0$ that represent their image under $\varphi$.

**2-skeleton:** Given $f: X' \to Y$, want to extend $f$ to a 2-cell $e^2$.

$e^2$ is attached to $X'$ along some loop $\gamma$.

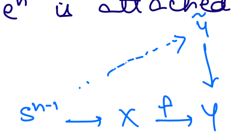
Thus $[\gamma] = 0$ in $\pi_1$, and thus $\varphi(\gamma) = 0$ in $\pi_1(Y, y_0)$. Thus the loop $f\gamma$ is nullhomotopic, and lifts to a loop in $\tilde{Y}$.




Now since $\tilde{Y}$ is contractible, we can extend the lift to a map $e^2 \to \tilde{Y}$, and thus to $e^2 \to Y$.

**3-skeleton & beyond:** Suppose we have $f: X^{n-1} \to Y$, $n \geq 3$. Want to extend $f$ to an $n$-cell $e^n$.

$e^n$ is attached to $X^{n-1}$ along $\partial e^n = S^{n-1}$




Since $S^{n-1}$ is simply-connected, can lift $S^{n-1} \to X \xrightarrow{f} Y$ to $S^{n-1} \to \tilde{Y}$. $\tilde{Y}$ contractible $\Rightarrow$ can extend to $e^n \to \tilde{Y}$, and hence to $e^n \to Y$.

- **Injectivity:** Suppose $f_0, f_1 : (X, x_0) \to (Y, y_0)$ satisfy $(f_0)_* = (f_1)_*$. Want to construct basept-preserving homotopy from $f_0$ to $f_1$.

  As before will exploit CW-structure of $X$ and construct homotopy skeleton by skeleton.

  **1-skeleton:** Choose maximal tree $T$ of $X'$.
  By collapsing each $f_i(T)$ to $y_0$ in $Y$, we can homotope each $f_i$ to maps sending $T \to y_0$. So assume $T \to y_0$ under both maps.
  Any leftover edge $e$ of $X'$ gets mapped by both $f_0, f_1$

to a loop representing the same element of $\pi_1(Y, y_0)$, so $f \circ e$ can be homotoped to $f \circ e$.

2-skeleton & beyond:

Suppose we've constructed homotopy $X^{n-1} \times I \cup X \times \{0,1\} \to Y$. Want to extend homotopy to $e^n \times I$, where $e^n$ is an $n$-cell of $X$. Note that $e^n \times I$ is an $(n+1)$-cell of $X \times I$. This is attached to $X^{n-1} \times I \cup X \times \{0,1\}$ via a map $S^n$, $n \geq 2$. As in the proof of surjectivity, we can lift $S^n \to X \to Y$ to $S^n \to \tilde{Y}$, and use this to extend our map to $e^n \times I \to Y$.

Remark: Both halves of this proof that involved extending a map into $Y$ from a 2-skeleton (or higher) to a 3-skeleton (or higher) really proved a general fact:

If $X$ is a CW-complex, then any map $f: X^2 \to BG$ can be extended to $X \to BG$.

This makes sense because maps into $BG$ are entirely coded on the level of $\pi_1$'s, and $\pi_1$'s are entirely captured by 2-skeletons.