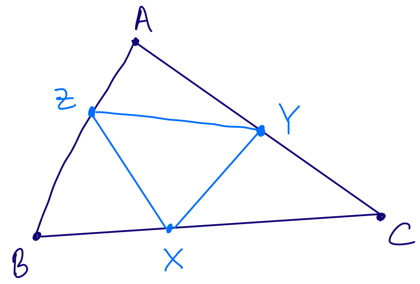
The Problem :

Given an acute angled triangle ABC.

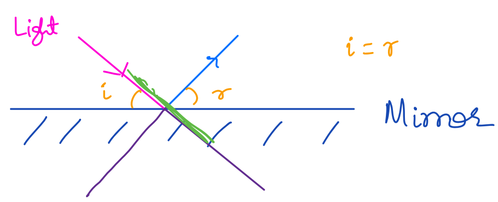


Pick points X, Y, Z on sides BC, CA, AB respectively.
When is the perimeter of ΔXYZ minimised?

It turns out that the perimeter is minimised when ΔXYZ is the orthic triangle.

In this talk we shall explore the wonderful world of reflections to uncover an unexpected solution to this problem, which will lead us to explore an idea recurring in studying billiard trajectories.

→ Reflections



A reflection can thus be seen as a "continuation of the pink line" inside the mirror
(Remember the "light takes the least distance path" from physics?)

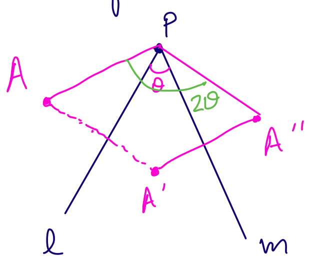
→ foray into Euclidean Isometries

- Reflections } "orientation - reversing"
- Translations
- Rotations } "orientation - preserving"

2 Orientation preservings ⇝ orientation - preserving

2 Orientation reversings ⇝ orientation - preserving

- Reflection about line l : R_l



$R_m \circ R_l$ = Rotation about P

- Composition of an odd no. of reflections (with some additional conditions) $\rightsquigarrow$ Reflection + Translation

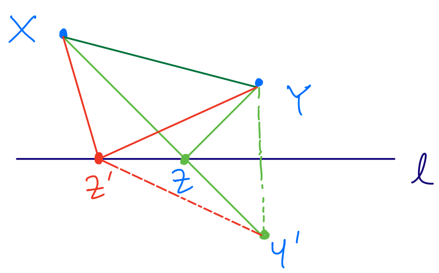
---

<u>Building up to the solution</u> :

Restated problem : Given 3 lines l, m, n, no 2 parallel, find points X, Y, Z on l, m, n resp so that perimeter of ΔXYZ minimised.

<u>Warm-Up 1</u>

Given a line l and 2 points X, Y on one side of it. Find (the unique) point $Z \in l$ so that perimeter ΔXYZ is minimised.



<u>Solution</u> : The length XY is fixed, so we really just need to minimise $XZ + YZ$.

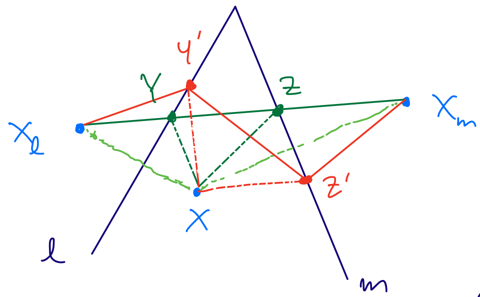
Reflect Y about l – let the reflection be Y'.

The point where XY' and l intersect does the job.

Note that $XZ + YZ = XZ + Y'Z = XY'$.
If we pick any other point Z' on l, then $XZ' + YZ' = XZ' + Y'Z'$ gives a "broken line" joining X and Y'. So we need to use $XY' \cap l$.

Warm-Up 2

Given 2 lines l, m and a point X not on either line, find $Y \in l$ and $Z \in m$ so that the perimeter ΔXYZ is minimised.



Solution: By Warm-Up 1, if we had a second point Y, we'd know that we'd get Z by reflecting X about m to get X_m, and joining $X_m Y$. In this case our perimeter would be equal to $XY + X_m Y$.

So we need to find Y on l so that $XY + X_m Y$ is minimised.

But this is the same as solving Warm-Up 1 - we have a line l and 2 points X, X_m on one side of it. So we know we should reflect X about l to get X_l, and join $X_m X_l$. Its intersection with l will give Y.

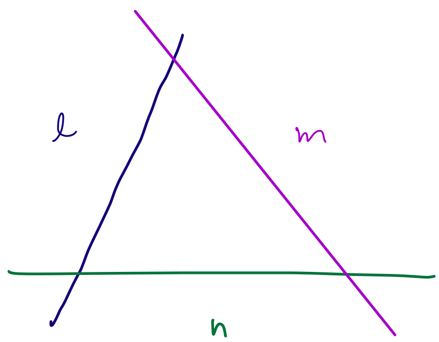
So, our final solution is :

Reflect X about l and m - let these reflections be X_l and X_m. Join $X_l X_m$ — the points at which it intersects l, m are our desired Y and Z.

As before, in this case the perimeter of XYZ can be visualised as the length of the segment $X_l X_m$.
And if we make any other choice Y', Z', then the perimeter $XY'Z'$ gives a "broken line" joining X_l and X_m.

Warm-Up 3 — Our Problem!

Given 3 lines l, m, n so that no 2 are parallel, find X, Y, Z on l, m, n resp. so that perimeter XYZ is minimised. (Note how this restates our original problem).

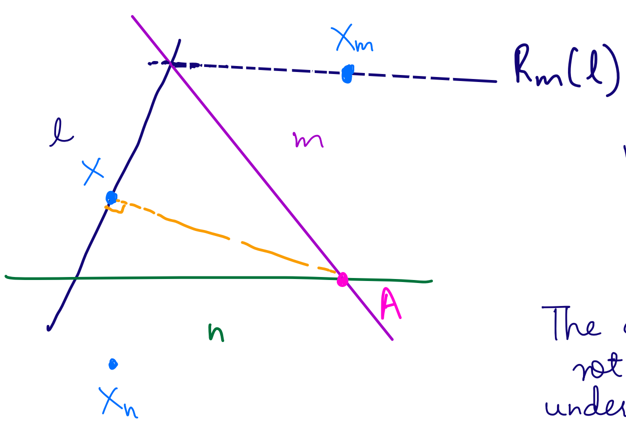


Solution: Let $R_l(P)$ denote reflection of P about line l.

By Warm-Up 2, if we had a point $X \in l$, we'd know what to do — reflect X about m, n, and join the points $R_m(X)$ and $R_n(X)$. The length of this segment would give us our perimeter.

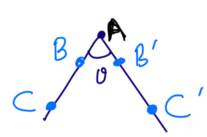
So we need to find $X \in l$ so that $R_m(X) R_n(X)$ is minimised. Note that X_m, X_n always lie on $R_m(l)$, $R_n(l)$.

Alternatively, let $R_m(X) = X_m$. Then $R_n(X) = R_n(R_m(X_m))$. So we want to find X_m on the reflected line $R_m(l)$ so that $X_m R_n \circ R_m(X_m)$ is minimised.



We know that $R_n \circ R_m$ is a rotation about $m \cap n = A$ (say) about a fixed angle.

The closer a point is to the center of rotation, the closer it is to its image under the rotation.



So we need X_m to be as close to A as possible — so we should drop a perpendicular from A to $R_m(l)$ to get X_m.

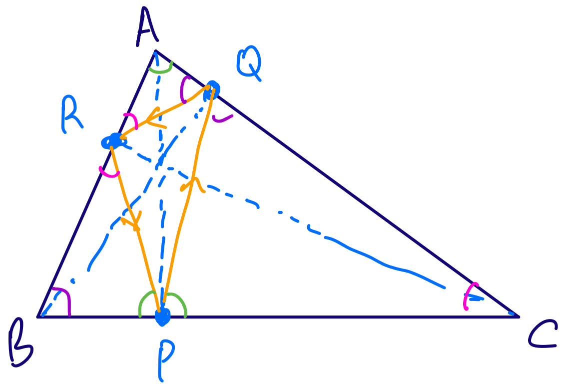
This amounts to dropping a perpendicular from A to l to get X.

Now we can proceed as before. Turns out the points Y, Z at which X_mX_n meets m, n resp. are also the foot of corresponding perpendiculars.

So there we have it! A way to get to this unexpected solution. Let's now see an alternative picture of how this puzzle is working, which will lead us to a recurring idea in billiards, as well as a way to generalise this problem to odd-sided polygons in general.

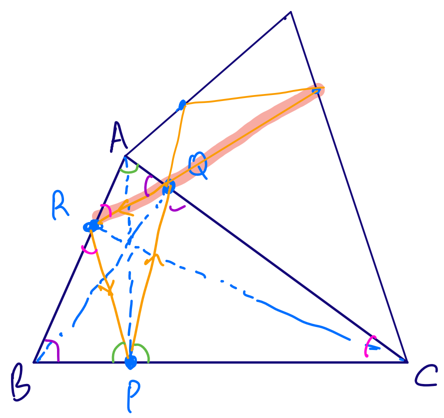
→ The Orthic Triangle

The Orthic triangle (i.e. the triangle formed by the feet of the three perpendiculars in a triangle) satisfies a cool angle property, as demonstrated in the triangle below:



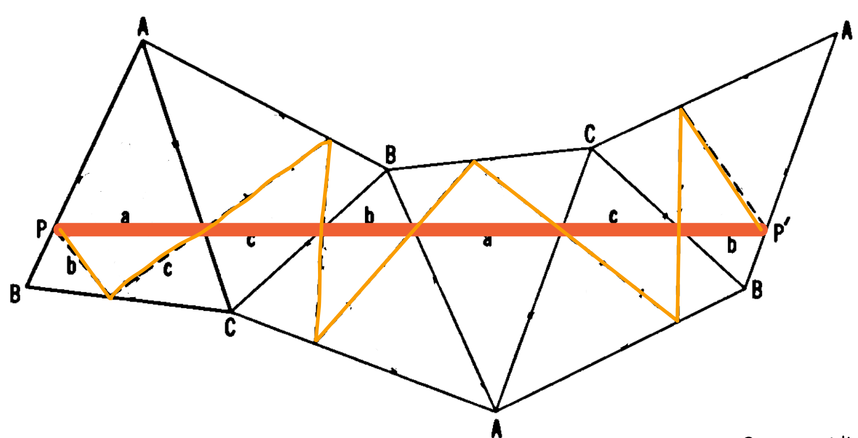
We can interpret this as – if you bounce a ray of light along one of the orange lines, it keeps circulating in this same orbit.

If we reflect / "unfold" ΔABC about one of its edges, as below,



the orange orthic triangle "unfolds" in a straight line, as highlighted in red above.

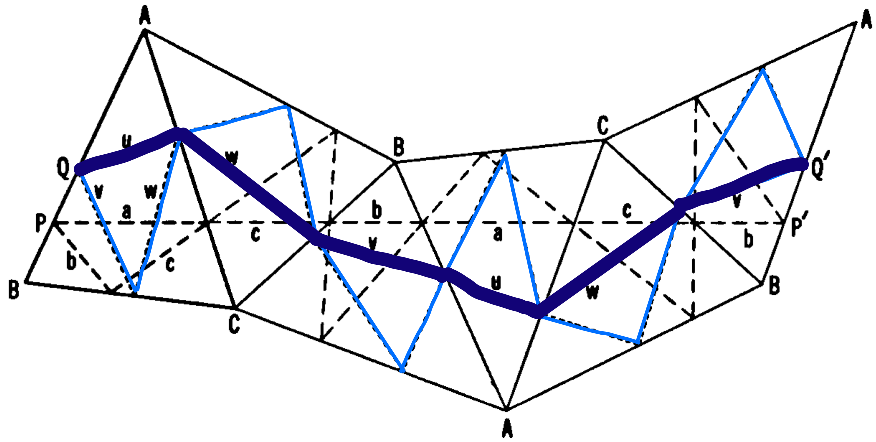
If we keep doing this, we get something like:



(Picture Credits: Coxeter & Greitzer, "Geometry Revisited")

Here we've successively reflected the triangle 6 times about each of its sides in turn — it is a fact that the rightmost side AB will always be a translation of the original side AB.

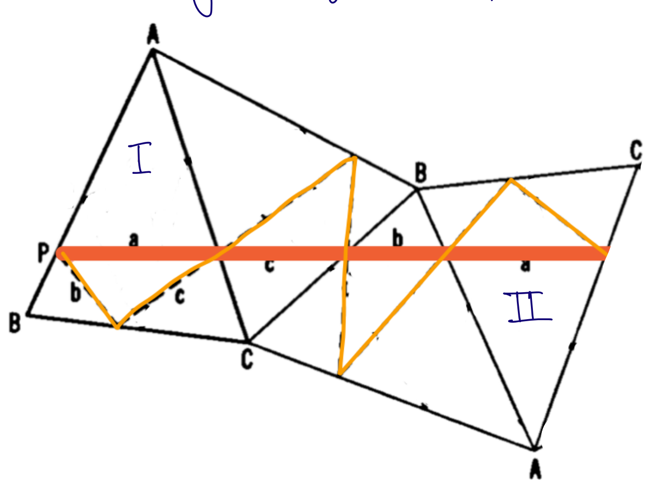
We see in the picture that the orange orthic triangle unfolds to a straight line joining the two parallel edges AB, whereas any other triangle would unfold to give a "broken line", as shown below in blue:



↳ This idea of reflecting or "unfolding" a surface is often used to study billiard trajectories.

As a final observation, let's see how to generalise this problem of finding a minimal perimeter inscribed polygon, to any odd-sided $(2n+1)$-gon, instead of just 3-gons:

Note the picture we get if we perform 3 instead of 6 reflections:

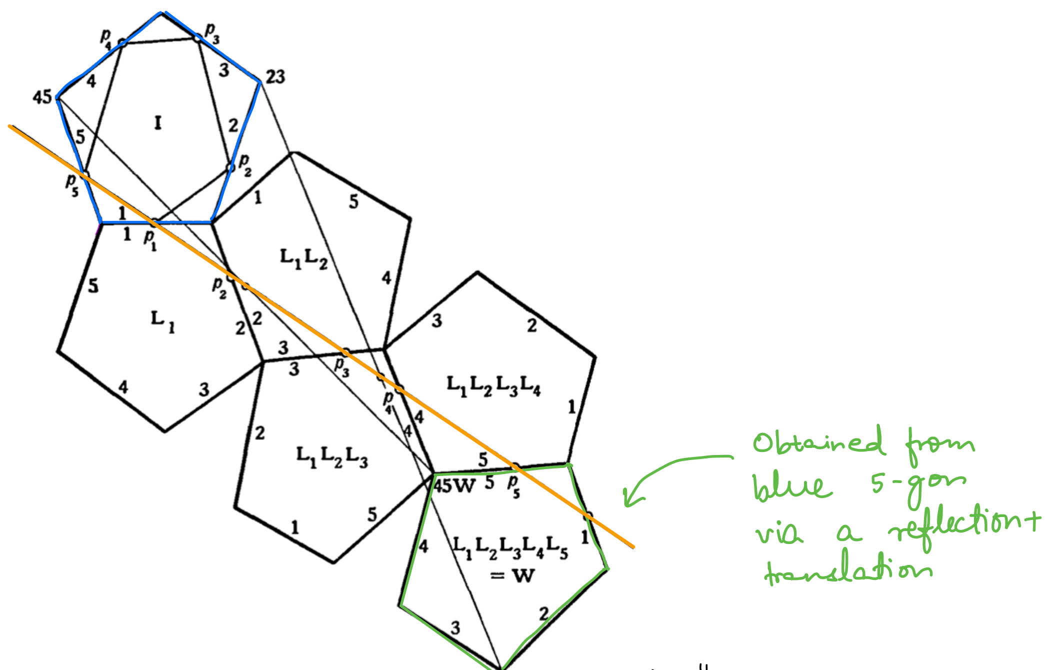


We've performed 3 reflections, so we know triangle II (above) is obtained from I by a reflection + translation.

The dark orange line gives this axis of reflection.

Turns out that the axis of reflection gives us the inscribed triangle of minimal perimeter (which coincides with the orthic triangle here).

We can use this idea of the axis of reflection to generalise our problem to $(2n+1)$-gons, as shown below:



Picture Credits: Morley, "Inversive Geometry"

Here, we have a 5-gon (in blue), and have reflected it about each of its sides in succession. The end result is a 5-gon obtained via a reflection + translation from the original.

The line in orange marks the axis of this reflection, and tracing it gives us the inscribed 5-gon of minimal perimeter.