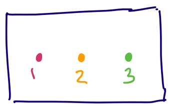
Configurations, Graphs and Trees

"The Homology of the Little Disks Operad" – Dev Sinha

Goal: To understand $\mathrm{Conf}_n(\mathbb{R}^d)$, specifically its H_* and H^*.

$\mathrm{Conf}_n(\mathbb{R}^d) := \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}^d,\ x_i \neq x_j\}$

I. $H_*(\mathrm{Conf}_n(\mathbb{R}^2))$



A point in $\mathrm{Conf}_3\mathbb{R}^2$

- $H_1(\mathrm{Conf}_n(\mathbb{R}^2)) \rightsquigarrow$ "1-dim structure"
  view loops as "particle dances"

Eg:

View this as a map

$S^1 \longrightarrow \mathrm{Conf}_n\mathbb{R}^2$

So get induced map

$H_1(S^1) \longrightarrow H_1(\mathrm{Conf}_n\mathbb{R}^2)$

- $H_2(\mathrm{Conf}_n\mathbb{R}^2) \rightsquigarrow$ "2-dim structure"

Eg:

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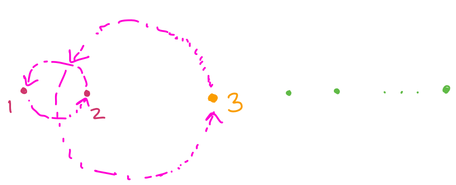


View these as maps

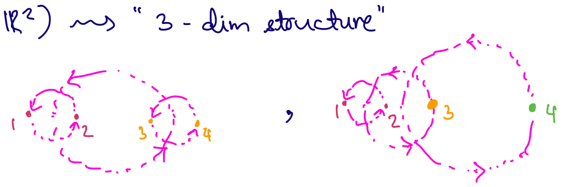
$S^1 \times S^1 \longrightarrow \mathrm{Conf}_n\mathbb{R}^2$

We get induced

$H_2(S^1 \times S^1) \longrightarrow H_2(\mathrm{Conf}_n\mathbb{R}^2)$

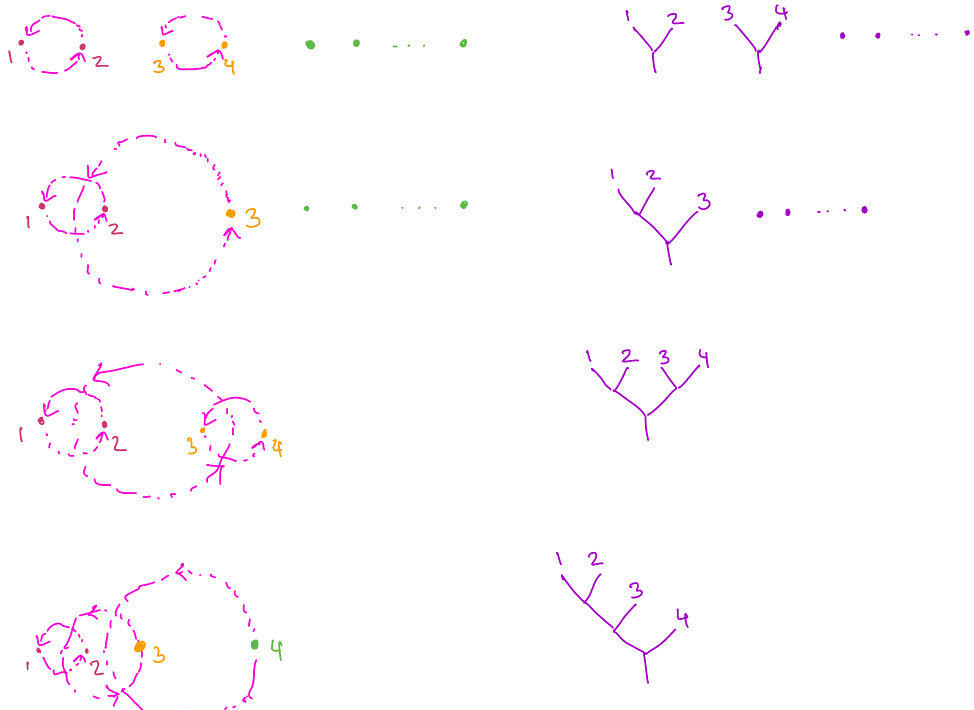
- $H_3(\mathrm{Conf}_n\mathbb{R}^2) \rightsquigarrow$ "3-dim structure"

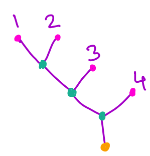
Eg:



$S^1 \times S^1 \times S^1 \longrightarrow \mathrm{Conf}_n\mathbb{R}^2$

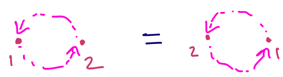
So we're constructing homology classes via "orbiting planet systems".
We can represent these orbiting systems using trees:



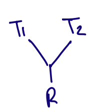
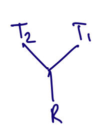
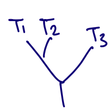
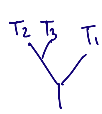
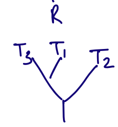


- internal vertices $\rightsquigarrow$ # internal vertices $|T|$ gives the homology degree
- root vertex
- leaves $\rightsquigarrow$ # leaves gives the no. of particles.

So given a (labelled) forest, we have an associated homology class.
What about relations b/w these classes?

Eg: 

($x \mapsto -x$ on S^1 is orientation preserving)

-  $-\ (-1)^{|T_1||T_2|}$  $= 0$ (Anti-Symmetry)
-  +  +  $= 0$ (Jacobi)

Rmk: In general, it is usually easier to describe generators of configuration spaces than their relations.

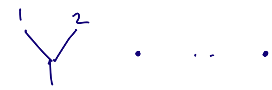
II $H_*(\mathrm{Confn}\ \mathbb{R}^d)$

$d > 2$

What changes? → Loops collapse (for eg in $\mathbb{R}^3$, we have enough dimensions to collapse the loop )

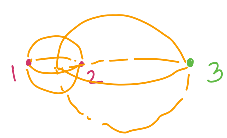
→ Don't have any homology until degree $d-1$

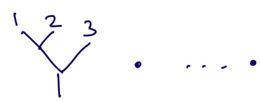
Eg $d = 3$



$S^2 \longrightarrow \mathrm{Confn}\ \mathbb{R}^3$

$H_2(S^2) \longrightarrow H_2(\mathrm{Confn}\ \mathbb{R}^3)$





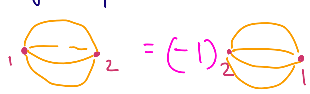
$S^2 \times S^2 \longrightarrow \mathrm{Confn}\ \mathbb{R}^3$

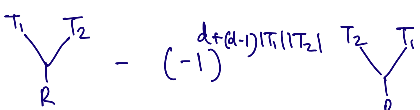
$H_4(S^2 \times S^2) \longrightarrow H_4(\mathrm{Confn}\ \mathbb{R}^3)$

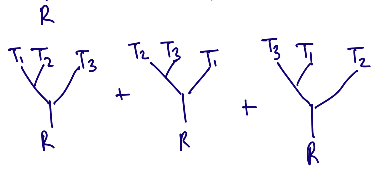
→ Can still use trees & forests to represent these classes

→ A tree T gives a degree $|T|(d-1)$ homology class

→ Anti-symmetry depends on d:

 ($x \mapsto -x$ on S^2 reverses orientation)

•  (Anti-Symmetry)

•  $= 0$ (Jacobi)

$*$

Defn: $\mathrm{Pois}^d(n) := \langle$ free module spanned by n-forests $\rangle / \{$ relations $(*)\}$

Rmk: $\mathrm{Pois}^2(n)$, $\mathrm{Pois}^3(n)$, $\mathrm{Pois}^4(n)$, ... differ in the anti-symmetry relation.

So we have a map

$$\mathrm{Pois}^d(n) \xrightarrow{\cong} \bigoplus H_k(\mathrm{Confn}\ \mathbb{R}^d) = H_*(\mathrm{Confn}\ \mathbb{R}^d)$$

Thm: This is an iso.

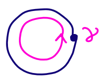
III Cohomology & Graphs

- $\omega \in H^1(\mathrm{Conf}_n \mathbb{R}^d)$

  i.e.

  $\omega : H_1(\mathrm{Conf}_n \mathbb{R}^d) \to \mathbb{Z}$

- $H^1(S^1)$:



S^1

γ : generates $H_1(S^1)$

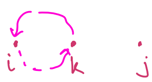
Have $\omega \in H^1(S^1)$ sending

$\gamma \mapsto 1$

Analogous cohomology class in $\mathrm{Conf}_n \mathbb{R}^2$:

$\alpha_{ij} \in H^1(\mathrm{Conf}_n \mathbb{R}^2)$

α_{ij} () $= 1$



α_{ij} () $= 0$

α_{ij} "extracts the motion of i wrt j"

Rigorously, we have a map

$$a_{ij} : \mathrm{Conf}_n \mathbb{R}^2 \longrightarrow S^1$$

$$(x_1, \dots, x_n) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

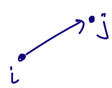
ω: top cohomology generator in $H^1(S^1)$

$$\boxed{\alpha_{ij} := a_{ij}^*(\omega)}$$

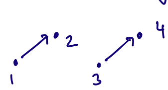
In general,

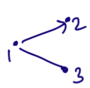
$$a_{ij} : \mathrm{Conf}_n \mathbb{R}^d \to S^{d-1}$$

$$\alpha_{ij} := a_{ij}^*(\omega) \qquad \omega \in H^{d-1}(S^{d-1})$$

- We represent α_{ij} as 

- Multiply cohomology classes by overlaying their graphs.

$\alpha_{12}\alpha_{34}$ 

$\alpha_{12}\alpha_{13}$ 

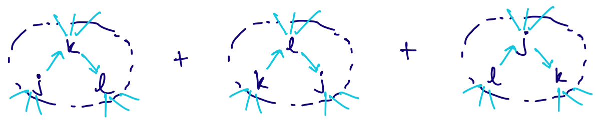
Rmk: Implicit here is an ordering σ of the edges of our graph G.

- # edges in graph $|E(G)| \rightsquigarrow$ cohomology in deg $|E(G)|(d-1)$

Relations :

- If G_1, G_2 differ in reversal of k arrows and edge reordering by σ,

$$G_1 - (-1)^{k(d-1)}(\operatorname{sgn}\sigma)^d G_2 = 0$$

-  $= 0$

Defn : $\mathrm{Siop}^d(n) :=$ free module on n-graphs / relations

$$\mathrm{Siop}^d(n) \xrightarrow{\cong} H^*(\mathrm{Conf}_n \mathbb{R}^d)$$

Thm : This is an iso

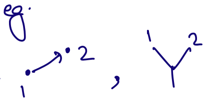
IV Graph-Tree Pairing

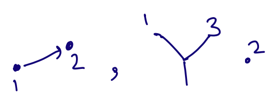
We have a homology - cohomology pairing

$$\langle \underset{H^k}{\omega}, \underset{H_k}{h} \rangle = \omega(h)$$

We can combinatorially develop an analogous pairing b/w graphs and trees.

We'll want, for eg:

$\langle$  $\rangle = 1$

$\langle$  $\rangle = 0$

Here's how we define the general pairing :

$$\langle G, T \rangle$$

$\beta_{G,T} : \{\text{edges of } G\} \to \{\text{internal vertices of } T\}$

 $\mapsto$ lowest vertex of path from i to j.

Eg:

$$\langle G, T\rangle := \begin{cases} 0 & \text{if } \beta_{G,T} \text{ is not a bijection} \\ (-1)^{f(d,\sigma)} & \text{if } \beta_{G,T} \text{ is a bijection} \end{cases}$$

Theorem: The combinatorial pairing $\langle G, T\rangle_{comb}$ is equivalent to the topological pairing $\langle G, T\rangle_{top}$.

How this helps:

- Can combinatorially find bases for $Pois^d(n)$, $Siop^d(n)$ (using the relations) and show $\langle \, , \, \rangle_{comb}$ is a non-degenerate pairing.
- Thus we get bases for H^*, H_* of $Conf_n \mathbb{R}^d$.
- Good prototypical examples to understand the phenomenon of rep. stability.