The Nerve Lemma & Spectral Sequences

(I) The Nerve Lemma

<u>Recall</u>: If $X = U \cup V$, U, V open

Then we have a Mayer-Vietoris long exact sequence:

$$\cdots \to H_i(U \cap V) \to H_i(U) \oplus H_i(V) \to H_i(X) \to H_{i-1}(U \cap V) \to \cdots$$

We can derive the following as applications:

(1) $U, V, U \cap V$ contractible $\Rightarrow \tilde{H}_i(X) = 0 \ \forall i$

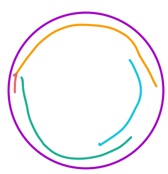
(2) If $X = U_1 \cup U_2 \cup U_3$ s.t. U_i, $U_i \cap U_j$, $U_1 \cap U_2 \cap U_3$ • nonempty • contractible

$$\Rightarrow \tilde{H}_i(X) = 0 \ \forall i$$

(3) $X = U_1 \cup U_2 \cup U_3$ s.t. U_i, $U_i \cap U_j$ contractible

$U_1 \cap U_2 \cap U_3 = \phi$

$\Rightarrow H_1(X) = \mathbb{Z}$, $H_0(X) = \mathbb{Z}$



$H_i(X) = 0$ o/w

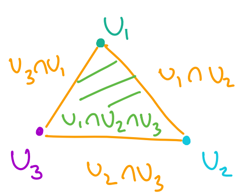
(same homology as S^1)

<u>Motivating the Nerve Lemma</u>:

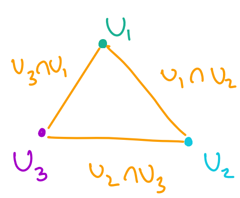
In each of the examples above, let's construct a simplicial complex, called the "<u>nerve</u>", associated to each covering:

(1)

(2)



(3)



In each case above, p-simplices $\leftrightarrow$ nonempty intersections of $p+1$ sets

<u>Note</u>: In each case, the <u>homology of X matches w/ the nerve of the covering</u>.

This holds in far more generality:

Nerve Lemma: Let X be a CW-complex, and X_α subcomplexes s.t.

- $X = \bigcup_{\alpha \in \Lambda} X_\alpha$
- Every simplex σ of X lies in finitely many X_α
- Every finite intersection $X_{\alpha_1} \cap \dots \cap X_{\alpha_n}$ is empty or contractible

Then $H_i(X) \cong H_i(\mathcal{N}(X_\alpha)) \quad \forall i$

Nerve of the covering

(II) Proof Motivation via Double Complexes

Let's start by recalling the proof of Mayer-Vietoris.

We have short exact sequences:

$$0 \leftarrow C_n(X) \leftarrow C_n(U) \oplus C_n(V) \leftarrow C_n(U \cap V) \leftarrow 0$$

$$(\sigma, -\sigma) \longleftarrow\!\shortmid \ \sigma$$

$$\tau + \nu \longleftarrow\!\shortmid \ (\tau, \nu)$$

These commute w/ the boundary maps, and so give us a short exact sequence of chain complexes:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
C_n(X) & \leftarrow & C_n(U) \oplus C_n(V) & \leftarrow & C_n(U \cap V) \\
\downarrow & & \downarrow & & \downarrow \\
C_{n-1}(X) & \leftarrow & C_{n-1}(U) \oplus C_{n-1}(V) & \leftarrow & C_{n-1}(U \cap V) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

This then gives us the Mayer-Vietoris LES (for eg, by using the Snake Lemma)

Note: • We showed the "horizontal homologies" of the above diagram are 0

• The Mayer-Vietoris LES involves the "vertical homologies"

For the Nerve Lemma, we will analyse the following double complex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
0 & C_2(X) \arrow[l] \arrow[d] & & \bigoplus C_2(X_\alpha) \arrow[ll] \arrow[d] & & \bigoplus C_2(X_\alpha \cap X_\beta) \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
0 & C_1(X) \arrow[l] \arrow[d] & & \bigoplus C_1(X_\alpha) \arrow[ll] \arrow[d] & & \bigoplus C_1(X_\alpha \cap X_\beta) \arrow[ll] \arrow[d] & \cdots \arrow[l] \\
0 & C_0(X) \arrow[l] & & \bigoplus C_0(X_\alpha) \arrow[ll] & & \bigoplus C_0(X_\alpha \cap X_\beta) \arrow[ll] & \cdots \arrow[l]
\end{tikzcd}$$

where: • vertical maps are the boundary maps on each space

• horizontal maps are similar to the ones we constructed for the Mayer-Vietoris LES.

More specifically, $\sigma \in C_n(X_{\alpha_0} \cap \ldots \cap X_{\alpha_k})$ gets mapped to $(\sigma, -\sigma, \sigma, \ldots, (-1)^k \sigma)$, viewed as an element of

$$\bigoplus_{i=0}^{k} C_n(X_{\alpha_0} \cap \ldots \cap \hat{X}_{\alpha_i} \cap \ldots \cap X_{\alpha_k})$$

One can check that the horizontal homologies of the above complex are always 0 (regardless of the topology of the X_α)

Thus if we replace the above complex w/ a 2-D array of groups comprising of the horizonal homologies, we get:

$$\begin{array}{cccc} \vdots & \vdots & \vdots & \\ C_2(X) & 0 & 0 & \cdots \\ \downarrow & & & \\ C_1(X) & 0 & 0 & \cdots \\ \downarrow & & & \\ C_0(X) & 0 & 0 & \cdots \end{array} \qquad — (*)$$

In the setting of the Nerve Lemma, we are additionally assuming each nonempty finite intersection is contractible, i.e. the vertical homologies vanish.

Thus if we replace our double complex with the vertical homologies at each position, we get :

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \cdots \\ \bigoplus_{X_\alpha \neq \phi} \mathbb{Z} & \leftarrow & \bigoplus_{X_\alpha \cap X_\beta \neq \phi} \mathbb{Z} & \leftarrow & \bigoplus_{X_\alpha \cap X_\beta \cap X_\gamma \neq \phi} \mathbb{Z} & \leftarrow & \cdots \end{array} \qquad — (**)$$

Key Observation : The bottom row is precisely the chain complex for the nerve $N(X_\alpha)$. (!!)

It is natural to want the homologies of both the chain complexes $(*)$ & $(**)$ to be the same.

Turns out - a simple spectral sequence argument implies the same.

(III) Spectral Sequences (for a double complex)

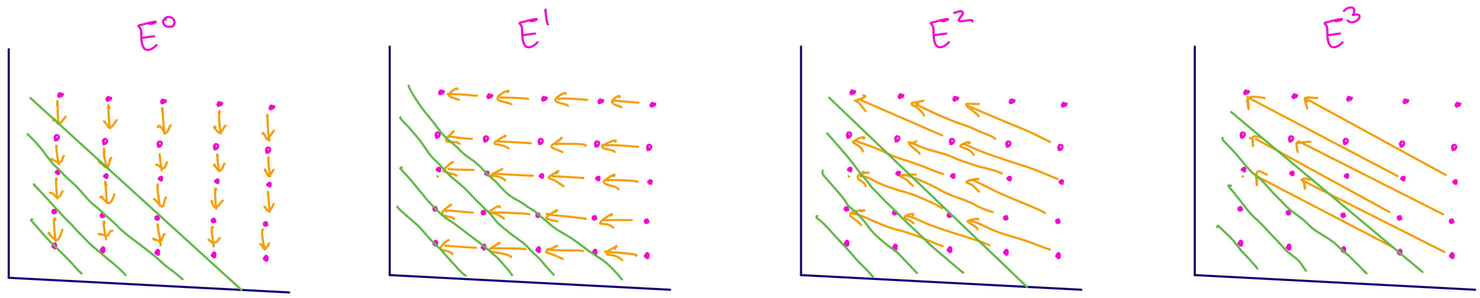
→ Think of as a sequence of "pages" $E^0, E^1, E^2, \ldots,$ each page with a 2D array of groups.

→ E^r_{pq} : group in position (p,q) on page r, $p, q \geq 0$.

→ Each page has a differential d^r s.t. $(d^r)^2 = 0$.

→ $E^{r+1}_{pq} = \dfrac{\text{ker of } d^r \text{ at } E^r_{pq}}{\text{im of } d^r \text{ at } E^r_{pq}}$ (in general, more data needed to determine d^r)

$\rightarrow$ $d^r : E^r_{pq} \rightarrow E^r_{p-r,\, q+r-1}$ n^{th} diag to $(n-1)^{th}$ diag



$\rightarrow$ For a given (p,q), we eventually have $E^r_{pq} = E^{r+1}_{pq} = \cdots$ for $r >> 0$.

Denote these stable groups as E^∞_{pq}.

Now,

- Suppose we have a double complex $C_{p,q}$.
- We can form a chain complex, called the "total complex" $(TC)_\bullet$, s.t. $(TC)_n = \bigoplus_{p+q=n} C_{p,q}$
- There is a spectral sequence associated to $C_{\bullet,\bullet}$.
  $\boxed{E^0_{pq} = C_{pq}}$, and the differentials d^0 are the vertical maps $C_{pq} \rightarrow C_{p\, q-1}$
- Thus $\boxed{E^1_{pq} \text{ is the "vertical homology" of } C_{\bullet,\bullet}}$.
  The differentials d^1 are induced from the horizontal maps $C_{pq} \rightarrow C_{p-1\, q}$
- The stable groups E^∞_{pq} on the n^{th} diagonal come from a filtration of $H_n((TC)_\bullet)$.
  Specifically, $\exists$
  $$0 \subseteq F^n_0 \subset F^n_1 \subset \cdots \subset F^n_n = H_n((TC)_\bullet)$$
  s.t. $E^\infty_{i,\, n-i} = F^n_i / F^n_{i-1}$

* In particular, if all but one $E^\infty_{p\, n-p}$ is 0, then that group equals $H_n((TC)_\bullet)$ *

Apply this now to the double complex constructed in the Nerve Lemma.

$$
\begin{tikzcd}
\vdots & \vdots & \\
\bigoplus C_1(X_\alpha) \arrow[d] & \bigoplus C_1(X_\alpha \cap X_\beta) \arrow[d] & \cdots \\
\bigoplus C_0(X_\alpha) & \bigoplus C_0(X_\alpha \cap X_\beta) & \cdots
\end{tikzcd}
$$

E^0

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \\
\bigoplus_{X_\alpha \neq \phi} \mathbb{Z} & \bigoplus_{X_\alpha \cap X_\beta \neq \phi} \mathbb{Z} \arrow[l] & \arrow[l] \cdots
\end{tikzcd}
\Big\}\ \text{chain complex of } \mathcal{N}(X_\alpha)
$$

E^1

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
H_0(\mathcal{N}) & H_1(\mathcal{N}) & H_2(\mathcal{N}) \arrow[llu]
\end{tikzcd}
$$

E^2

<u>Note</u>:
- $E^2_{pq} = E^\infty_{pq}$
- $E^2_{pq} = 0$ if $q > 0$ $\quad (= E^\infty_{pq})$
- $E^2_{p0} = H_p(\mathcal{N})$ $\quad (= E^\infty_{p0})$

Thus $\boxed{H_*(\mathcal{N}) \cong H_*((TC)_\bullet)}$

We can now apply the same trick to the <u>transpose of $C_{\bullet,\bullet}$</u>.

$$
\begin{tikzcd}
\vdots & \vdots & \\
\bigoplus C_0(X_\alpha \cap X_\beta) \arrow[d] & \bigoplus C_1(X_\alpha \cap X_\beta) \arrow[d] & \cdots \\
\bigoplus C_0(X_\alpha) & \bigoplus C_1(X_\alpha) & \cdots
\end{tikzcd}
$$

E^0

$$
\begin{tikzcd}
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
C_0(X) & C_1(X) \arrow[l] & C_2(X) \arrow[l]
\end{tikzcd}
$$

E^1

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \\
H_0(X) & H_1(X) & H_2(X) \arrow[llu] & \cdots \arrow[llu]
\end{tikzcd}
$$

E^2

By the same reasoning,

$$\boxed{H_*(X) \cong H_*((TC)_\bullet)}$$

Thus $\boxed{H_*(X) \cong H_*((TC)_\bullet) = H_*(\mathcal{N})}$