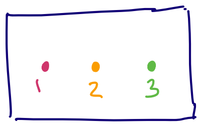
Configurations, Graphs and Trees

"The Homology of the Little Disks Operad" – Dev Sinha

<u>Goal</u>: To understand $\mathrm{Conf}_n(\mathbb{R}^2)$, specifically its H_* and H^*.

$$\mathrm{Conf}_n(\mathbb{R}^2) := \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}^2,\ x_i \neq x_j\}$$



A point in $\mathrm{Conf}_3 \mathbb{R}^2$

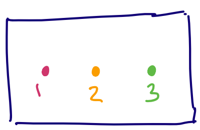
<u>Why Study this?</u> (A biased answer)

Helps understand:

- Braid Group (Co)homology
- Homological / Representation Stability
- $H_k(\mathrm{Conf}_n \mathbb{R}^d)$

Today we'll focus on concrete combinatorial descriptions of H_* and H^*.

(I) <u>$H_*(\mathrm{Conf}_n(\mathbb{R}^2))$ and Trees</u>



A point in $\mathrm{Conf}_3 \mathbb{R}^2$

- $H_1(\mathrm{Conf}_n(\mathbb{R}^2)) \rightsquigarrow$ "1-dim structure"
  view loops as "particle dances"

<u>Eg</u>:

View this as a map

$$S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$$

So get induced map

$$\underset{\underset{\mathbb{Z}}{\wr\wr}}{H_1(S^1)} \longrightarrow H_1(\mathrm{Conf}_n \mathbb{R}^2)$$

We can use this strategy to study higher-degree H_k.

- $H_2(\mathrm{Conf}_n \mathbb{R}^2) \rightsquigarrow$ "2-dim structure"

  We can construct maps

  $$S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$$

  And look at the (images of) induced map

  $$\underset{\substack{\rotatebox{90}{$\cong$} \\ \mathbb{Z}}}{H_2(S^1 \times S^1)} \longrightarrow H_2(\mathrm{Conf}_n \mathbb{R}^2)$$

Eg:

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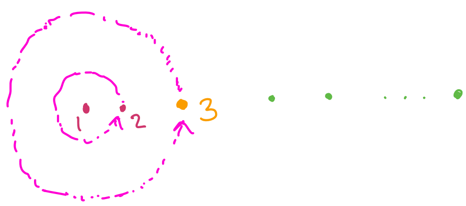
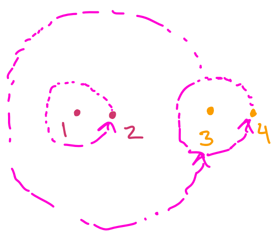


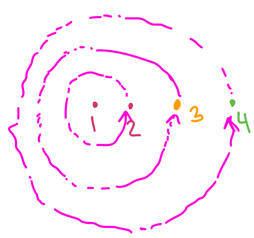
- $H_3(\mathrm{Conf}_n \mathbb{R}^2)$

  $$S^1 \times S^1 \times S^1 \longrightarrow \mathrm{Conf}_n \mathbb{R}^2$$

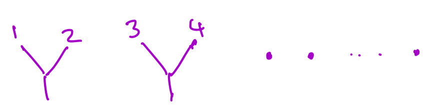
Eg:

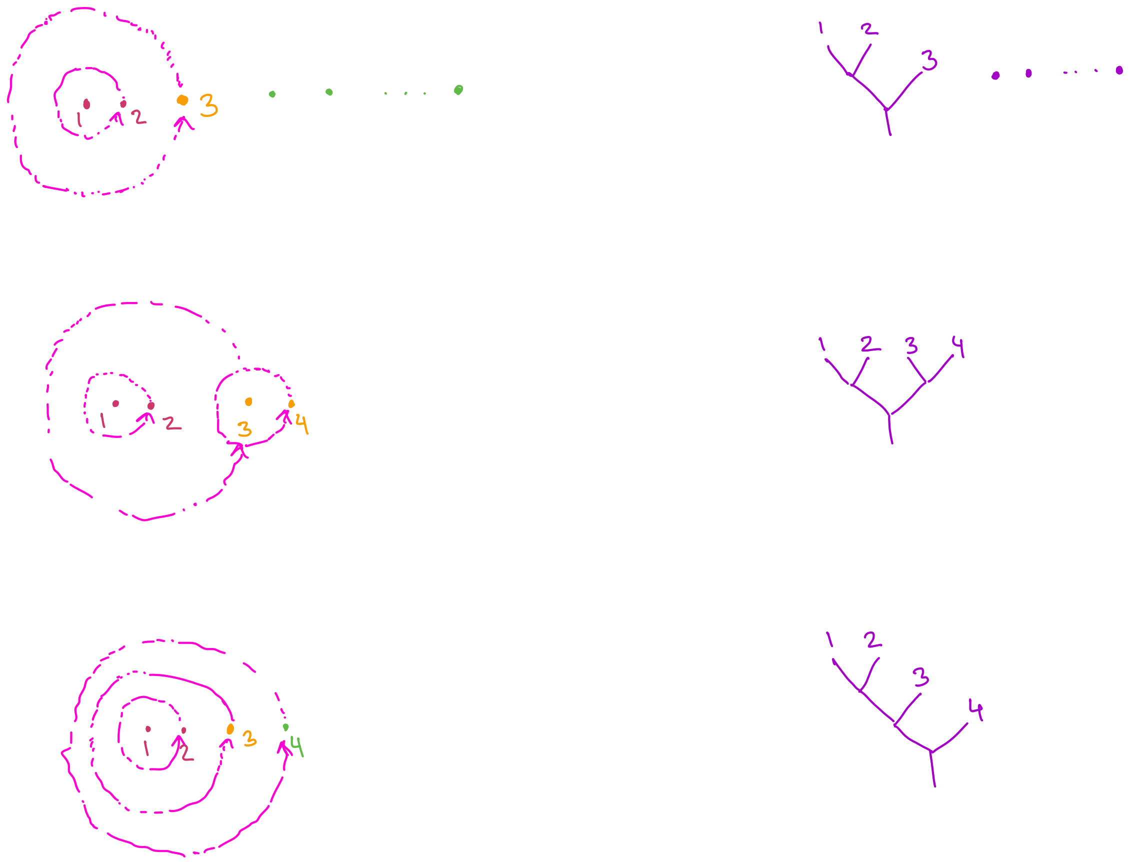


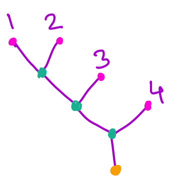
Eg:



So we're constructing homology classes via "orbiting planet systems".
We can represent these orbiting systems using trees:



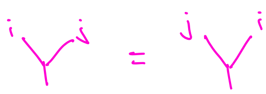




- internal vertices $\rightsquigarrow$ # internal vertices $|T|$ gives the homology degree
- root vertex
- leaves $\rightsquigarrow$ # leaves gives the no. of particles.

So given a (labelled) forest, we have an associated homology class.

<u>Note</u>:
- We have so far only described <u>some</u> homology gp. elements. We can get others by, for eg, taking linear combinations of them.
- Turns out, the above described classes <u>generate</u> all the homology. But they have some relations between them, i.e. they are not a basis.

<u>Eg</u>: 

(II) <u>Cohomology & Graphs</u>

How do we start thinking about H^*?

<u>Just for the purposes of this talk</u>, $H^k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{Z})$

(In general, we always have a natural surjection $H^k \to \mathrm{Hom}(H_k, \mathbb{Z})$, which is an iso if, for eg, all H_k's are torsion-free)

So for eg. $H^1(S^1) \cong \mathrm{Hom}(\underset{\underset{\mathbb{Z}}{\text{S}||}}{H_1(S^1)}, \mathbb{Z}) \cong \mathbb{Z}$

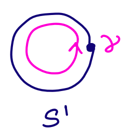
To study $H^1(\mathrm{Conf}_n \mathbb{R}^2)$, we'll construct maps

$$\mathrm{Conf}_n \mathbb{R}^2 \to S^1$$

to get induced

$$\underset{\underset{\mathbb{Z}}{\text{S}||}}{H^1(S^1)} \to H^1(\mathrm{Conf}_n \mathbb{R}^2)$$

- $\omega \in H^1(S^1)$



γ : generates $H_1(S^1)$
Have $\omega \in H^1(S^1)$ sending $\gamma \mapsto 1$

Analogous cohomology class in $H^1(\mathrm{Conf}_n \mathbb{R}^2)$:

$\alpha_{ij} \in H^1(\mathrm{Conf}_n \mathbb{R}^2)$

α_{ij} "extracts the motion of i wrt j"

$\alpha_{ij}($  $) = 1$

$\alpha_{ij}($  $\cdot j) = 0$

Rigorously, we have a map

$$a_{ij} : \mathrm{Conf}_n \mathbb{R}^2 \to S^1$$

$$(x_1, \ldots, x_n) \mapsto \frac{x_i - x_j}{|x_i - x_j|}$$

ω : Generator of

$$\boxed{\alpha_{ij} := a_{ij}^*(\omega)}$$

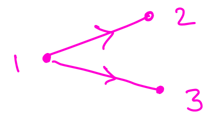
Represent α_{ij} by a (directed) graph 

So,
- we've described some elements of $H^1(\mathrm{Conf}_n \mathbb{R}^n)$
- turns out, they also generate H^1
- we understand their action on H_1, as $H_1 \to \mathbb{Z}$

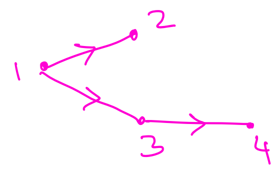
→ What about higher H^k?

Using the cohomology cup product, we can get some higher-degree cohomology classes.
We'll represent these as graphs too.

Eg: $\alpha_{12}\alpha_{13} \in H^2$



$\alpha_{34}\alpha_{12}\alpha_{13} \in H^3$



→ Implicit in these graphs is an ordering σ of the edges, to record the order of multiplication of the α_{ij}'s.

Note: # edges $|E(G)|$ ⇝ degree of cohomology $|E(G)|$

→ But how can we understand these graphs as $\mathrm{Hom}(H_k, \mathbb{Z})$?

III Graph-Tree Pairing

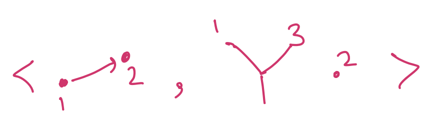
So far, we have a way of associating cohomology classes to directed labelled graphs (with an ordering on edges)

To understand how they act on H_k, we need to unpack how the cup product works.

But it turns out, there is a combinatorial rule that captures this.

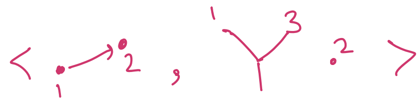
This should, for eg, give us:

 = 1

 = 0

Here's how we define the general pairing $\langle G, T \rangle$:

① If there's an edge $i \to j$ in G s.t. there is no path b/w i & j in T, then $\underline{\langle G, T \rangle = 0}$

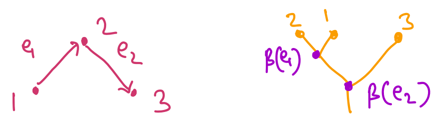
Eg :  $= 0$

② Otherwise, define

$\beta_{G,T} : \{ \text{edges of } G \} \to \{ \text{internal vertices of } T \}$

$i \to j \quad \mapsto \quad$ lowest vertex of path from i to j.

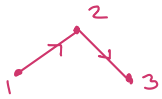
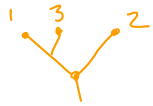
Eg:

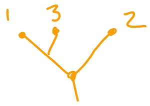


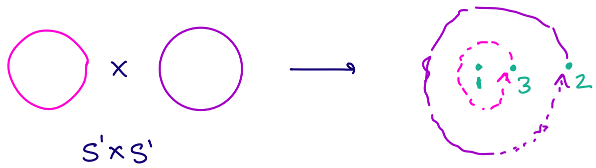
$$\langle G, T \rangle = \begin{cases} 0 & \text{if } \beta_{G,T} \text{ not a bijection} \\ \pm 1 & \text{o/w} \end{cases}$$

depends on σ and T

Aside (Not for the Talk): Why the rule works

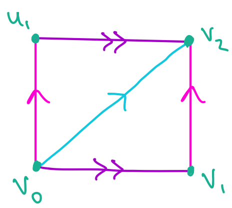
Eg: $\alpha_{12}\alpha_{23}$ =  , 

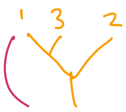
Recall:  is obtained from the image of the map

F : 

S'xS'

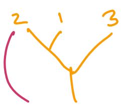
Give a Δ-complex structure to S'xS':

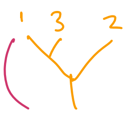


Then $\alpha_{12}\alpha_{23}$ () $= \alpha_{12}\alpha_{23}\big(F[v_0 u_1 v_2] + F[v_0 v_1 v_2]\big)$

$$= \underbrace{\alpha_{12}(F[v_0u_1])}_{0}\alpha_{23}(F[u_1u_2]) + \alpha_{12}(F[v_0v_1])\underbrace{\alpha_{23}(F[v_1v_2])}_{0}$$

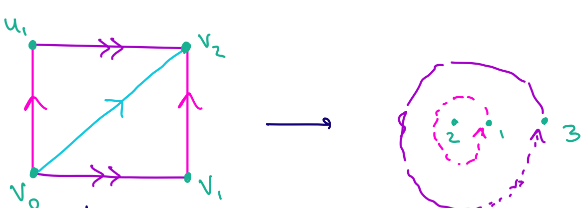
$$= 0$$

On the other hand, for $\alpha_{12}\alpha_{23}$ (), the analogous computation yields:

$\alpha_{12}\alpha_{23}$ () $= \alpha_{12}\alpha_{23}\big(F[v_0 u_1 v_2] + F[v_0 v_1 v_2]\big)$

$$= \underbrace{\alpha_{12}(F[v_0u_1])}_{1}\underbrace{\alpha_{23}(F[u_1u_2])}_{1} + \underbrace{\alpha_{12}(F[v_0v_1])}_{0}\alpha_{23}(F[v_1v_2]) = 1$$

where

F : 

S'xS'

In general, we can give $(S^1)^k$ a Δ-complex structure by partitioning $[0,1]^k$ into $k!$ k-simplices of the form $[v_0\, v_1 \dots v_k]$, where each $v_i v_{i+1}$ is an edge of $[0,1]^k$.

Thus for $\omega \in H^k$, $h \in H_k$, $\omega(h)$ is a sum of terms of the form $\alpha_{i_0 i_1}([v_0, v_1])\, \alpha_{i_1 i_2}([v_1, v_2]) \dots \alpha_{i_{k-1} i_k}([v_{k-1}\, v_k])$.

The image of each edge $[v_j\, v_{j+1}]$ is a single S^1-orbit in the homology class. Note that orbits correspond to internal vertices of the associated tree

Thus, a given term is ± 1 iff $[v_j\, v_{j+1}]$ maps to an orbit s.t. particles i_j and i_{j+1} are on different components of that orbit.

This can happen iff the pairs (i_j, i_{j+1}) — which correspond to the edges of the graph G — can be put in bijection with the orbits — corresponding to internal vertices of T — s.t. each pair i_j, i_{j+1} is on different components of the associated orbit. This is exactly what the combinatorial rule said.

All other terms will be forced to be 0, and thus $\omega(h) \neq 0$ iff a bijection of the above form exists.