An Introduction to Group (Co)homology

(I) Group (Co)Homology, Algebraically

G : Group

$\mathbb{Z}G$: Group Ring finite sums $\sum n_i g_i$, $n_i \in \mathbb{Z}$, $g_i \in G$

$\mathbb{Z}G$-module : Abelian group with a G-action.
Can view $\mathbb{Z}$ as a $\mathbb{Z}G$-module with trivial G-action.

Take a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\cdots \to F_n \xrightarrow{\partial_n} \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

- exact
- F_i free over $\mathbb{Z}G$

Rmk : • Free resolutions always exist
• Can make this definition using "projective" modules, in general.

Homology $H_*(G; M)$

Tensor with $\otimes_{\mathbb{Z}G} M$:

$$\cdots \to F_n \otimes_{\mathbb{Z}G} M \xrightarrow{\partial_n \otimes id} \cdots \to F_1 \otimes_{\mathbb{Z}G} M \xrightarrow{\partial_1 \otimes id} F_0 \otimes_{\mathbb{Z}G} M \longrightarrow 0$$

Take homologies :

$$H_n(G; M) := \frac{\ker \partial_n \otimes id}{\operatorname{im} \partial_{n+1} \otimes id}$$

(Alternatively, $\boxed{H_*(G; M) := \operatorname{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}; M)}$)

Cohomology $H^*(G; M)$

Apply $\operatorname{Hom}_{\mathbb{Z}G}(-, M)$:

$$\cdots \leftarrow \operatorname{Hom}_{\mathbb{Z}G}(F_n, M) \xleftarrow{\delta_{n-1}} \cdots \xleftarrow{\delta_1} \operatorname{Hom}_{\mathbb{Z}G}(F_1, M) \xleftarrow{\delta_0} \operatorname{Hom}_{\mathbb{Z}G}(F_0, M) \leftarrow 0$$

Take cohomologies :

$$H^n(G; M) := \frac{\ker \delta_n}{\operatorname{im} \delta_{n-1}}$$

(Alternatively, $\boxed{H^*(G; M) := \operatorname{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}; M)}$)

Rmk: These definitions are independent of the choice of free (and more generally, projective) resolution.

$$\begin{tikzcd}
\cdots \arrow[r] & F_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] & \cdots \arrow[r] & F_1 \arrow[r, "\partial_1"] \arrow[d, shift right, "f_1"'] & F_0 \arrow[r, "\partial_0"] \arrow[d, shift right, "f_0"'] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & F_n' \arrow[r, "\partial_n'"] \arrow[u, shift right, "g_n"'] & \cdots \arrow[r] & F_1' \arrow[r, "\partial_1'"] \arrow[u, shift right, "g_1"'] & F_0' \arrow[r, "\partial_0'"] \arrow[u, shift right, "g_0"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The chain maps $f_\bullet$, $g_\bullet$ give a "chain homotopy equivalence".

Examples

(1) $G = \mathbb{Z}$

$\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$

$H_*(\mathbb{Z};\mathbb{Z})$

$$0 \longrightarrow \mathbb{Z}[t,t^{-1}] \xrightarrow[\text{by } (t-1)]{\text{multiply}} \mathbb{Z}[t,t^{-1}] \xrightarrow{\text{eval } t=1} \mathbb{Z} \to 0$$

$$\Big\downarrow \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

After $\otimes_{\mathbb{Z}[t,t^{-1}]}\mathbb{Z}$, t gets identified with 1, $(t-1) \mapsto (1-1) = 0$

$$\boxed{H_n(\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & o/w \end{cases}}$$

(2) $G = \mathbb{Z}/m\mathbb{Z}$

$\mathbb{Z}G \cong \dfrac{\mathbb{Z}[t]}{(t^m - 1)}$

$M := t^{m-1} + t^{m-2} + \cdots + t + 1$.

$M(t-1) = t^m - 1 = 0$

$H^*(\mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$

$$\cdots \xrightarrow{M} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{M} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow[\text{at } t=1]{\text{eval}} \mathbb{Z} \to 0$$

$$\Big\downarrow \mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$$

$$\cdots \xleftarrow{m} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{m} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

$$\boxed{H^n(\mathbb{Z}/m\mathbb{Z}\,;\,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n \text{ odd} \\ \mathbb{Z}/m\mathbb{Z} & n \text{ even} \end{cases}}$$

(II) Group (Co)Homology, Topologically

(1) Cellular (Co)homology ($\mathbb{Z}$ - coefficients)

X : CW - complex

C_n : free abelian gp. generated by n-cells

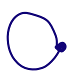
$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{aug} \mathbb{Z} \to 0$$

- $H_n(X;\mathbb{Z}) = \dfrac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$

- $H^{n+1}(X;\mathbb{Z}) = \dfrac{\ker \delta_n}{\operatorname{im} \delta_{n-1}}$ where δ_i come from the dual complex

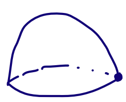
$$\cdots \xleftarrow{\delta_2} \mathrm{Hom}(C_2,\mathbb{Z}) \xleftarrow{\delta_1} \mathrm{Hom}(C_1,\mathbb{Z}) \xleftarrow{\delta_0} \mathrm{Hom}(C_0,\mathbb{Z}) \leftarrow 0$$

Eg:



$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Eg:



$$0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

(2) $K(G,1)$ Spaces

G - group

Fact : $\exists$ a topological space, called the $\boxed{K(G,1)\text{-space } X}$,

s.t.

$$\pi_1(X) \cong G$$

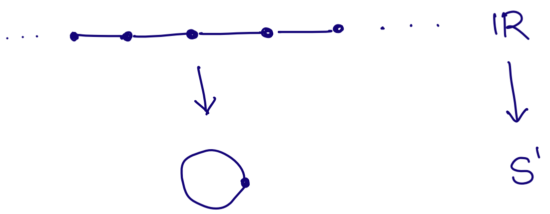
and $\tilde{X}$ univ. cover contractible
$$\downarrow$$
$$X$$

Fact : X is unique upto homotopy

Defn: $H_*(G;\mathbb{Z}) = H_*(X;\mathbb{Z})$, $H^*(G;\mathbb{Z}) = H^*(X;\mathbb{Z})$

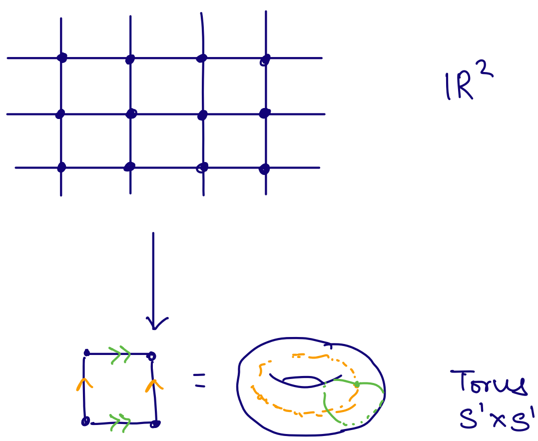
Rmk: A CW-complex $K(G,1)$ always exists.

Eg: $G = \mathbb{Z}$



$- K(\mathbb{Z},1)$

Eg: $G = \mathbb{Z}^2$

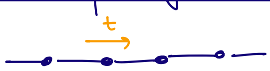


$- K(\mathbb{Z}^2,1)$

Equivalently: If $\tilde{X}$ contractible has a free, cellular G-action, then $X = \tilde{X}/G$ is a $K(G,1)$.

$H^*(G;\mathbb{Z}) \cong H^*(\tilde{X}/G;\mathbb{Z})$, $\quad H_*(G;\mathbb{Z}) \cong H_*(\tilde{X}/G;\mathbb{Z})$

(3) Equivalence of the algebraic and topological versions:

Running Example: $G = \mathbb{Z} \curvearrowright$  $\cdots$ $\mathbb{R}$

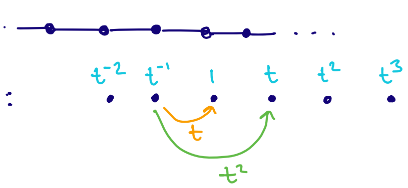
$\mathbb{Z}G \cong \mathbb{Z}[t,t^{-1}]$

Step 1: Suppose X is a CW-complex, contractible

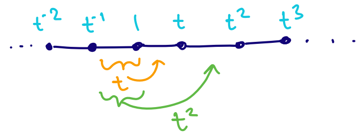
$G \curvearrowright X$ cellular action, free

Then each chain group $C_n(X)$ is an abelian gp w/ free G-action hence a free $\mathbb{Z}G$-module

Eg:



C_0 : $\cdots \cong \mathbb{Z}[t,t^{-1}]$

C_1:  $\cong \mathbb{Z}[t, t^{-1}]$

so the chain complex for the space is:

$$0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{t-1} \mathbb{Z}[t,t^{-1}] \to 0$$

<u>Step 2</u>: Since $\tilde{X}$ is contractible, the chain complex is <u>exact</u>. Hence, it gives us a <u>free resolution of $\mathbb{Z}$</u> (over $\mathbb{Z}G$)!

<u>Step 3</u>: <u>Note</u>: $C_n(\tilde{X}/G) \cong C_n(\tilde{X})/G$

<u>Eg</u>: $C_1(S^1) \cong \mathbb{Z} \cong \mathbb{Z}[t,t^{-1}]/(\text{action of } t) \cong C_1(\mathbb{R}^1)/(\text{action of } t)$

<u>Also</u>: $C_n(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z} \cong C_n(\tilde{X})/G$ (tensoring with the trivial module kills off G-action)

- So we started with a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\cdots \to C_2(\tilde{X}) \to C_1(\tilde{X}) \to C_0(\tilde{X}) \to 0$$

(For $G = \mathbb{Z}$, $\tilde{X} = \mathbb{R}$: $0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{t-1} \mathbb{Z}[t,t^{-1}] \to 0$)

- Applying $\otimes_{\mathbb{Z}G}\mathbb{Z}$ gives us the chain complex for $\tilde{X}/G$:

$$\cdots \to C_2(\tilde{X})\otimes_{\mathbb{Z}G}\mathbb{Z} \to C_1(\tilde{X})\otimes_{\mathbb{Z}G}\mathbb{Z} \to C_0(\tilde{X})\otimes_{\mathbb{Z}G}\mathbb{Z} \to 0$$ } Algebraic defn. of $H_*(G;\mathbb{Z})$

$$=$$

$$\cdots \to C_2(\tilde{X}/G) \to C_1(\tilde{X}/G) \to C_0(\tilde{X}/G) \to 0$$ } Topological defn of $H_*(G;\mathbb{Z})$

(For $G = \mathbb{Z}$, $\tilde{X}/G = S^1$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$)

Similarly, to prove the same for H^*, we use:

$$\begin{aligned}
\operatorname{Hom}_{\mathbb{Z}G}(C_n(\tilde{X}), \mathbb{Z}) &\cong \operatorname{Hom}_{\mathbb{Z}G}(C_n(\tilde{X}), \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})) \\
&\cong \operatorname{Hom}_{\mathbb{Z}}(C_n(\tilde{X}) \otimes_{\mathbb{Z}G} \mathbb{Z}, \mathbb{Z}) \\
&\cong \operatorname{Hom}_{\mathbb{Z}}(C_n(\tilde{X}/G), \mathbb{Z})
\end{aligned}$$

III Applications of having both Perspectives:

(1) Bounding cohomological dimension

If we can find a contractible $\tilde{X}$ with $G \curvearrowright \tilde{X}$ free + proper s.t. $\dim X = n$, then

$$cd(G) \leq n$$

i.e. G can't have any cohomology beyond deg n

(2) Other finiteness properties: FP_n, FP_∞

FP_n : $\exists$ a free resolution with $F_0, F_1, \ldots, F_n$ finitely gen. over $\mathbb{Z}G$

Topology: If we find a $K(G,1)$ with finitely many cells in dim $\leq n$, then G is type FP_n

FP_∞ : $\exists$ a free resolution with $F_0, F_1, \ldots, F_n, \ldots$ finitely gen. over $\mathbb{Z}G$

Topology: If we find a $K(G,1)$ with finitely many cells in every dim, then G is type FP_∞.

(3) Cohomology of (virtual) duality groups

Eg: $G = SL_n \mathbb{Z}$

(1) $H^k(SL_n\mathbb{Z}; \mathbb{Q}) = 0$ for $k > \binom{n}{2}$] Using Topology, via $SL_n\mathbb{Z} \curvearrowright X_n$ with finite stabilizers

(2) $H^k(SL_n\mathbb{Z}; \mathbb{Q}G) = 0$ $\forall$ $i \neq \binom{n}{2}$] Using Topology — Poincare duality on X_n

(3) Fact (2) implies:

$$H^{\binom{n}{2}-i}(SL_n\mathbb{Z}; \mathbb{Q}) \cong H_i(SL_n\mathbb{Z}; \underset{\text{"Steinberg module"}}{\underbrace{D}} \otimes_{\mathbb{Q}} \mathbb{Q})$$

] Algebraically