① **Defn:** A bundle of groups $p: E \to X$ with fibre $G$ is a covering space s.t. each fibre $p^{-1}(x)$ has a group structure iso. to $G$, and each

i.e. local trivialisations {$x \in X$ has an open nbhd $U$ s.t. $p^{-1}(U) \underset{\text{homeo}}{\cong} U \times G$, and this homeo is a gp. iso on each fibre (Important!)

② (Non) Examples: ① The covering $\mathbb{R} \to S^1$ is NOT a bundle of groups (though it is a principal $\mathbb{Z}$-bundle).

This is because even though we do have local trivialisations $p^{-1}(U) \cong U \times G$, we cannot arrange for them to always be a group iso on fibres ($n \mapsto n+1$ is not a gp iso on $\mathbb{Z}$).

In fact, will see soon that a bundle of gps $E \to X$ always has a copy of $X$ embedded in $E$

② For a manifold $M$, the orientation cover $M_{\mathbb{Z}} \to M$ (consisting of one copy of $M$ for $n=0$, and a copy of the double sheeted oriented cover $\tilde{M}$ for all other $n = \pm 1, \pm 2, \dots$) is a bundle of gps w/ fibre $\mathbb{Z}$.

The gp iso induced by the local trivialisations on each fibre is either $n \mapsto n$ or $n \mapsto -n$.

③ $E \to X$ will always have a copy of $X$: The map sending $x$ to the element of $p^{-1}(x)$ corresponding to the identity $1_G$ embeds $X$ homeomorphically in $E$.

④ Path lifts induce gp isos: Given a path $\gamma$ in $X$, with $\gamma(0) = x_0$, $\gamma(1) = x_1$, we get a map $p^{-1}(x_0) \to p^{-1}(x_1)$ that sends a point $\tilde{x}_0$ to $\tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the lift of $\gamma$ starting at $\tilde{x}_0$. This is a gp iso

(Break $\gamma$ into finitely many pieces $\gamma_1, \dots, \gamma_n$ s.t. each $\gamma_i$ lies in a trivialisable open set — each $\gamma_i$ then induces an iso b/w the fibres of its endpts, and hence so does $\gamma$)

In particular, $\pi_1(X, x_0)$ acts on $p^{-1}(x_0) \cong G$ by gp autos.

for eg: for $M_2 \to M$, if a loop in $M$ lifts to a loop in $\tilde{M}$, then it acts via the identity on the fibre. If it doesn't (i.e. if the loop is orientation-reversing), it acts via $n \mapsto -n$

⑤ For fixed $G$, $X$,

Bundles of gps $E \to X$ w/ fibre $G$ $\longleftrightarrow$ $\pi_1(X) \curvearrowright G$ via automorphisms, i.e. gp homs $\pi_1(X) \to \mathrm{Aut}(G)$

- Given an action of $\pi_1 X$ on $G$, can construct a bundle by quotienting $\tilde{X} \times G$ ($\tilde{X}$: univ. cover) by the diagonal action of $\pi_1$ (in general, can construct cover w/ fiber $F$ via $\tilde{X} \times F / \pi_1(X)$ in this way)
- Given a bundle $E \to X$, the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is "the same as" the action of $\pi_1(X, x_1)$ on $p^{-1}(x_1)$, i.e:

Take a path $\gamma$ from $x_0$ to $x_1$. Can think of $\pi_1(X, x_1)$ as $\gamma\, \pi_1(X, x_0)\, \gamma^{-1}$.

Then $\gamma$ induces an iso $p^{-1}(x_0) \to p^{-1}(x_1)$ equivariant under $\pi_1$-action, i.e. for each $h \in \pi_1(X, x_0)$,

$$\gamma \circ \underbrace{h}_{\in \pi_1(X,x_0)} = \underbrace{(\gamma h \gamma^{-1})}_{\in \pi_1(X,x_1)} \gamma \qquad (\text{as a map } p^{-1}(x_0) \to p^{-1}(x_1))$$

(i.e. $\gamma$ induces a $\pi_1 X$-equivariant iso b/w the fibres $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$)

Rmk: Above pf shows isos of bundles over $X$ correspond to $\pi_1(X)$-equivariant automorphisms $G \to G$.

## ⑥ Pullback bundles & bundle maps

6.1 Bundle map: $E \to X$, $E' \to X'$ bundles w/ fibre $G$

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

s.t. $\tilde{f}$ is a group iso on every fibre

6.2 Pullback bundle: Given $\begin{tikzcd} & E' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$, construct pullback $f^*(E') \to X$.

As a set, $f^*(E') = \{(x, e') : x \in X,\ e' \in p^{-1}(f(x))\}$

(i.e. using the fibre over $f(x)$ to construct the fibre over $x$.)

Use local trivialisations over $U' \subset X'$ to construct local trivialisations over $f^{-1}(U') \subset X$.

We have a natural bundle map

$$\begin{tikzcd} f^*E' \arrow[r, "\tilde{f}"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

## 6.3 Pullbacks give all bundle maps

Given a bundle map

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

the map $E \to f^*E'$ sending $e \mapsto (p(e), \tilde{f}(e))$ is an iso of bundles.

Thus pullbacks give all bundle maps

## 6.4 Homotopy and Pullbacks

Suppose we have $\begin{tikzcd} & E' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$ and $f^*E'$ is the pullback bundle.

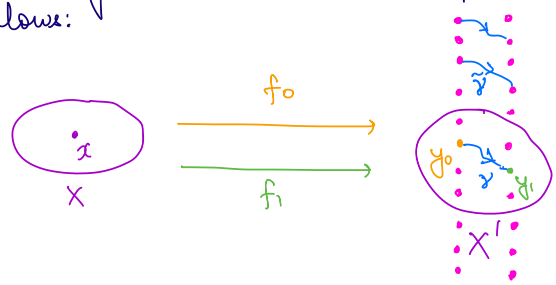
Consider $\gamma \in \pi_1(X, x_0)$, and the action of $\gamma$ on $p^{-1}(x_0) \subset f^*E'$.
This can be understood as follows: the fibre $p^{-1}(x_0)$ is, by construction, the fibre $(p')^{-1}(f(x_0))$. Consider the loop $f\gamma \in \pi_1(X', f(x_0))$.
The action of $f\gamma$ on $(p')^{-1}(f(x_0))$ matches up with the action of $\gamma$ on $p^{-1}(x_0)$. (break the path $\gamma$ up into locally trivialisable components)

On the level of $\pi_1$-modules, this corresponds to using the gp hom $f_* : \pi_1(X, x_0) \to \pi_1(X', f(x_0))$ to induce a $\pi_1 X$-module structure on the $\pi_1(X')$-module structure on $G$ corresponding to $E' \to X'$.

Thus pullback bundles correspond to induced $\pi_1$-module structure

Now suppose two maps $f_0, f_1 : X \to X'$ are homotopic. Then they induce the same gp homs on $\pi_1$'s, and thus $f_0^*E' \cong f_1^*E'$.

Concretely, this iso can be pictured geometrically as follows:




$\gamma$ is the path from $f_0(x) = y_0$ to $f_1(x) = y_1$ traced out by the homotopy.
Then the lifts $\tilde{\gamma}$ from the fibre over $y_0$ to the fibre over $y_1$ give the bundle isomorphism.

## ⑦ Bundles over $X$ w/ fibre $G$ $\longleftrightarrow$ maps upto homotopy $X \to B\mathrm{Aut}G$

(Ex. 3H.3 in Hatcher)

The statement:

$\mathcal{B}(X;G)$ := bundles of gps over $X$ w/ fibre $G$, upto isomorphism

$E_0 \to B\mathrm{Aut}G$ := bundle w/ fibre $G$ corresponding to the natural action of $\mathrm{Aut}\,G \curvearrowright G$.

$[X, Y]$ := homotopy classes of maps $f : X \to Y$.

Then the map

$$\begin{aligned} [X, B\mathrm{Aut}\,G] &\longrightarrow \mathcal{B}(X;G) \\ f &\longmapsto f^*E_0 \end{aligned}$$

is a bijection.

Proof:

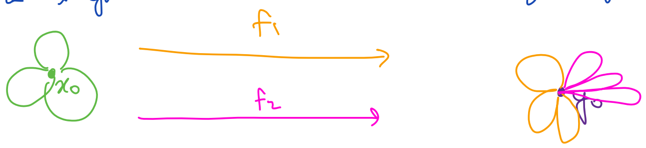
Surjectivity: given a bundle in $\mathcal{B}(X,G)$, it corresponds to a hom. $\pi_1(X, x_0) \to \mathrm{Aut}\,G$. Can construct a map $f : X \to B\mathrm{Aut}G$ that achieves this homomorphism on $\pi_1$'s, and then by tracing the construction of $f^*E_0$, can show that $f^*E_0$ is the desired bundle.

Injectivity : Suppose $f_1^*(E_0) \cong f_2^*(E_0)$. Pick $x_0 \in X$, let $y_1 = f_1(x_0)$, $y_2 = f_2(x_0)$. Assume for now that $y_1 = y_2 = y_0$.

$f_1^*(E_0) = f_2^*(E_0)$ means that both bundles correspond to isomorphic actions of $\pi_1(X, x_0)$ on $G$. i.e. $\exists$ an automorphism $\rho \in \operatorname{Aut} G$ s.t. for every $\gamma \in \pi_1(X, x_0)$, $\rho \circ (f_1)_*(\gamma) = (f_2)_*(\gamma) \circ \rho$. Thus $(f_2)_*(\gamma) = \rho\, ((f_1)_*\gamma)\, \rho^{-1} \quad \forall\, \gamma \in \pi_1(X, x_0)$.

Now we construct the homotopy.

First construct on $X'$. Contract a maximal tree $T$ of $X'$ to get wedge of circles — we know that $f_i|_X \simeq f_i|_{X/T}$, so can assume $X'$ has a single vertex $x_0$ and is a wedge of circles.




Now homotope the wedge under $f_1$ to the one under $f_2$ by moving the centre of the wedge along a loop corresponding to $\rho \in \operatorname{Aut} G$ (this is where we use $\rho \circ (f_1)_*\gamma \circ \rho^{-1} = (f_2)_*\gamma$)


To extend homotopy from $X' \times I$ to $X \times I$, proceed inductively. Suppose have homotopy on $X^{n-1} \times I$, $n \geq 2$. This is the $n$-skeleton of $X \times I$. Take an $(n+1)$-cell $e^{n+1}$, attached by its bdry $\partial e^{n+1} \cong S^n$. Since $S^n$ simply connected, can lift attaching map to $S^n \to E\operatorname{Aut} G$, and since $E\operatorname{Aut} G$ contractible, this allows us to extend to $e^{n+1} \to E\operatorname{Aut} G$, and thus to $e^{n+1} \to E\operatorname{Aut} G$.

Now, to deal with the general $y_1 \neq y_2$ case, note that we can always homotope $f_1$ to a map sending $x_0 \mapsto y_2$: Homotope on $X'/T$ by simply elongating the wedge along a path from $y_1$ to $y_2$, then extend to $X \times I$ by the same method described above.

Remark : The last part of the proof didn't use anything special about this problem's context, and really proved a general fact: If $X$ is a CW-complex, then any map $f : X^2 \to BG$ can be extended to $X \to BG$.

This makes sense because maps into $BG$ are entirely coded on the level of $\pi_1$'s, and $\pi_1$'s are entirely captured by 2-skeletons.

Related useful fact : $[X, Y] :=$ homotopy classes of maps $X \to Y$

$\langle X, Y \rangle :=$ base pt.-preserving homotopy classes of maps $X \to Y$

When $X, Y$ are path-connected,

$$[X, Y] = \langle X, Y \rangle / \pi_1(Y)$$