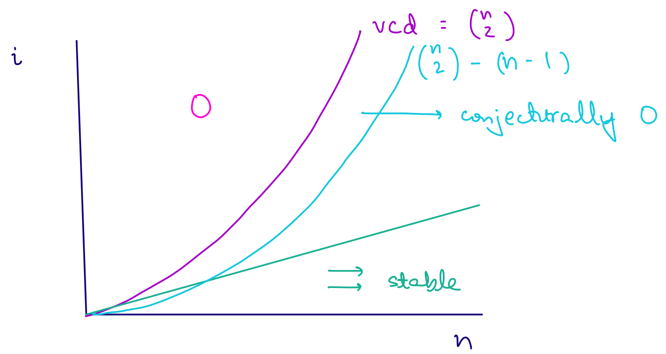
① **Motivation**: Calculating $H^*(SL_n\mathbb{Z};\mathbb{Q})$

$H^i(SL_n\mathbb{Z};\mathbb{Q})$



- $SL_n\mathbb{Z}$ is a "rational duality group", which we can use to study H^* close to the vcd.
- We have an analogous picture for SL_nR for number rings R, $Sp_{2n}R$, etc. (quadratic vcd, stable H^*, duality).

Unknown: Highest $q=q(n)$ s.t. $H^q(SL_n\mathbb{Z};\mathbb{Q})\neq 0$

Conjecture (Church-Farb-Putman): $H^{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q})\cong 0$ for $i\leq n-2$

Progress:
- True for
  - $i=0$ (Lee-Szczarba)
  - $i=1$ (Church-Putman)
  - $i=2$ (Brück-Miller-Patzt-Sroka-Wilson)
- Highest deg non-zero class found in deg $\binom{n}{2}-(n-1)$ (Ash, 2024)

Goal of the Talk:
- Duality Statement, Steinberg Modules
- Applications
- Borel-Serre, background on symmetric space

II Duality Statement & The Steinberg Module

Thm: $H^i(SL_n\mathbb{Z};\mathbb{Q}) \cong H_{\binom{n}{2}-i}(SL_n\mathbb{Z};\mathbb{Q}\otimes_{\mathbb{Z}} St_n\mathbb{Q})$

Steinberg Module

The Tits Building $\mathcal{T}_n(\mathbb{Q})$

Vertices $\longleftrightarrow$ $0 \subsetneq V \subsetneq \mathbb{Q}^n$ proper, nonzero subspaces

p-simplices $\longleftrightarrow$ Flags of subspaces

$$0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_p \subsetneq \mathbb{Q}^n$$

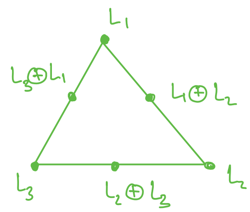
$\dim \mathcal{T}_n\mathbb{Q} = n-2$

Eg: $n=2$. $\mathcal{T}_2\mathbb{Q}$ is a 0-dim complex, with a vertex for every line $L \subset \mathbb{Q}^2$.

Suppose $\mathbb{Q}^2 = L_1 \oplus L_2$

"apartment"

Eg: Suppose $\mathbb{Q}^3 = L_1 \oplus L_2 \oplus L_3$ is a frame. Then the following is a subcomplex of $\mathcal{T}_3\mathbb{Q}$:



"apartment"

In general, for a frame $\mathbb{Q}^n = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, we get an apartment in $\mathcal{T}_n\mathbb{Q}$, which is iso to the barycentrically subdivided boundary of an $(n-1)$-simplex.

Thus an apartment is $\cong S^{n-2}$, and gives an element in $H_{n-2}(\mathcal{T}_n\mathbb{Q})$, called an "apartment class"

Thm: [Solomon-Tits] $\mathcal{T}_n\mathbb{Q} \simeq \vee S^{n-2}$

$St_n\mathbb{Q} := \tilde{H}_{n-2}(\mathcal{T}_n\mathbb{Q};\mathbb{Z})$ is generated by apartment classes.

(but NOT a basis)

Rmk: • $SL_n\mathbb{Z} \curvearrowright T_n\mathbb{Q}$, so $St_n\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}[SL_n\mathbb{Z}]$-module.

• We can similarly define $T_n\mathbb{F}$ and $St_n\mathbb{F}$ for any field $\mathbb{F}$. When R is a number ring and $\mathbb{F}$ its field of fractions, we have an analogous duality result for $SL_n R$ in terms of $St_n\mathbb{F}$.

(III) Computing $H^*(SL_n\mathbb{Z};\mathbb{Q})$

$$H^i(SL_n\mathbb{Z};\mathbb{Q}) \cong H_{\binom{n}{2}-i}(SL_n\mathbb{Z}; St_n)$$

• flat resolution of $\mathbb{Q}[SL_n\mathbb{Z}]$-modules

$$\cdots \to P_1 \to P_0 \to St_n \to 0$$

• Apply $\otimes_{SL_n\mathbb{Z}}\mathbb{Q}$ (or take coinvariants), and take H_*

$$\cdots \to (P_1)_{SL_n\mathbb{Z}} \to (P_0)_{SL_n\mathbb{Z}} \to 0$$

$$\left((P_i)_{SL_n\mathbb{Z}} := P_i / \langle m - gm : m \in P_i,\ g \in SL_n\mathbb{Z} \rangle \right)$$

In particular,

- can use partial resolutions for H_* in low degrees
- $H_0(SL_n\mathbb{Z}; St_n) \cong (St_n)_{SL_n\mathbb{Z}} := St_n / \langle m - gm : m \in St_n,\ g \in SL_n\mathbb{Z} \rangle$

Eg: $(St_2)_{SL_2\mathbb{Z}}$

$$St_2 = \tilde{H}_0(T_2\mathbb{Q}) = \tilde{H}_0(\{\text{lines in } \mathbb{Q}^2\})$$

Generated by $L_1 - L_2$ for $\mathbb{Q}^2 = L_1 \oplus L_2$

$$M = \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}$$

Apply $A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \in SL_2\mathbb{Z}$

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}\right) = \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} - \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} = -M$$

So, in $(St_2)_{SL_2\mathbb{Z}}$, $[M] = -[M]$

$\therefore [M] = 0$

However the same trick doesn't work with

$\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}$

Problem: $\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix}$ is an "integral apt class",

$\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}$ is not

But notice:

$$\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix} = \left(\mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix} - \mathbb{Q}\begin{bmatrix}1\\1\end{bmatrix}\right) + \left(\mathbb{Q}\begin{bmatrix}1\\1\end{bmatrix} - \mathbb{Q}\begin{bmatrix}2\\3\end{bmatrix}\right)$$

↑ integral apartment classes ↑

Defn: An apartment class is integral if it arises from a frame of $\mathbb{Z}^n$.

Thm: $St_n\mathbb{Q}$ is generated by integral apartment classes

(and this allows us to prove $H^{\binom{n}{2}}(SL_n\mathbb{Z};\mathbb{Q}) \cong (St_n\mathbb{Q})_{SL_n\mathbb{Z}} = 0$)

Rmk: This is not true for $St_n\mathbb{F}$ for all number fields $\mathbb{F}$.

One way to construct partial resolutions

$$St_n\mathbb{Q} = \tilde{H}_{n-2}(\mathcal{T}_n\mathbb{Q}) \implies St_n\mathbb{Q} = \ker\left(C_{n-2}(\mathcal{T}_n\mathbb{Q}) \to C_{n-3}(\mathcal{T}_n\mathbb{Q})\right)$$

↑ dim (n-2)

To surject onto $St_n\mathbb{Q}$, we thus need to "fill in the dim(n-2) holes" in $\mathcal{T}_n\mathbb{Q}$.

One Idea: • Try attaching (n-1)-cells whose boundaries attach to these holes, and hope we can compute $SL_n\mathbb{Z}$-coinvariants of the (n-1)-chains.

↑ (With many detail omissions!)

- Then to further build on the resolution, will want to tack on n-dim cells, and so on.

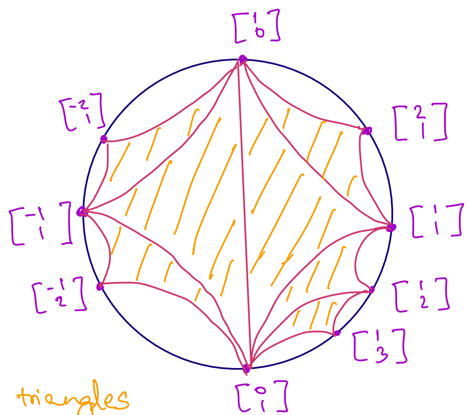
- Exactness of resolution $\longleftrightarrow$ vanishing H_* of augmented complex, which is implied by high connectivity.
  There are various combinatorial topology tools to prove high connectivity of simplicial complexes.

Eg: $n=2$.

$St_2 = \tilde{H}_0(\mathbb{Z}^2\mathbb{Q}) = \tilde{H}_0(\mathbb{Q} \cup \{\infty\})$

First augmentation: edges b/w vertices forming "integral basis"

$\rightarrow$ Farey Graph! $\leftarrow$



Second augmentation: fill in the triangles

(IV) Borel-Serre Duality

Defn: G is said to be a "rational duality group" if $\exists\ k \in \mathbb{N}$ and a $\mathbb{Q}G$-module D (dualising module) s.t. for every $\mathbb{Q}G$-module M,

$$H^i(G; M) \cong H_{k-i}(G; D \otimes_{\mathbb{Q}} \mathbb{Q})$$

Rmk: If the above holds, then D is uniquely determined, as

$$H^k(G; \mathbb{Q}G) \cong H_0(G; D \otimes_{\mathbb{Q}} \mathbb{Q}G) \cong (D \otimes \mathbb{Q}G)_G$$
$$:= D \otimes \mathbb{Q}G / \langle m - gm \mid g \in G \rangle$$
$$\cong D$$

So $\boxed{D \cong H^k(G; \mathbb{Q}G)}$

Thm [Bieri - Eckmann] : A group G (of type FP) is a rational duality group iff $H^i(G; \mathbb{Q}G) = 0 \;\; \forall \; i \neq k$

Fact : If $\tilde{X}$ is contractible and $\tilde{X}/G$ cpct (and $\tilde{X} \to \tilde{X}/G$ is a cover), then $H^i(G; \mathbb{Q}G) \cong H^i_c(\tilde{X}; \mathbb{Q})$

Defn : Symmetric Space

$SL_n\mathbb{Z} \curvearrowright SL_n\mathbb{R}/SO(n) = X_n$ "symmetric space"

$g \cdot A = gAg^T$

$\dim X_n = \binom{n+1}{2} - 1$

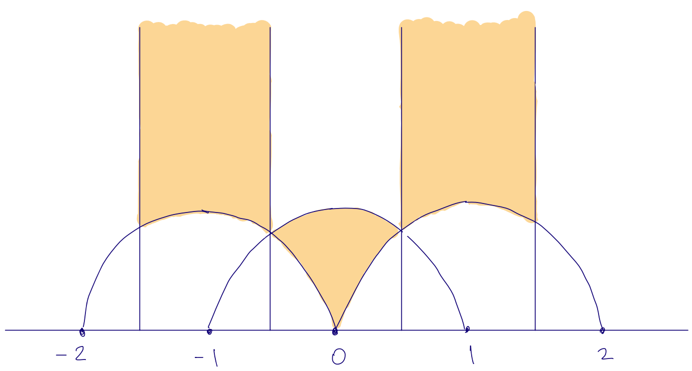
- X_n is contractible
- Action not free. Finite stabilisers (not a problem for $\mathbb{Q}$-coeffs)

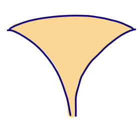
Problem : $X_n / SL_n\mathbb{Z}$ is not compact

Eg : When $n = 2$,

$SL_2\mathbb{Z} \curvearrowright SL_2\mathbb{R}/SO_2 \cong \mathbb{H}^2$

$SL_2\mathbb{R} \curvearrowright \mathbb{H}^2$ by fractional linear maps



$X_2 / SL_2\mathbb{Z}$:  not compact

But if we add all the "corner" pts, we'll compactify X_2 to $\overline{X}_2$ without changing its homotopy type.

(Note : Corners $\longleftrightarrow$ $\mathbb{Q} \cup \{\infty\}$ $\longleftrightarrow$ lines in $\mathbb{Q}^2$ $\longleftrightarrow$ vertices in $\mathcal{T}_2\mathbb{Q}$)

Thm [Borel - Serre] There exists a compactification $X_n \subset \overline{X}_n$ s.t.:

- $X_n \hookrightarrow \overline{X}_n$ is a homotopy equivalence, $SL_n\mathbb{Z} \curvearrowright X_n$ extends to $\overline{X}_n$.
- $\overline{X}_n / \Gamma$ is compact
- $\overline{X}_n$ is a topological (not smooth) manifold of dim $= \binom{n+1}{2} - 1$
- $\partial \overline{X}_n \simeq \mathcal{T}_n \mathbb{Q}$ "Tits building" - is a simplicial complex, $\simeq \vee S^{n-2}$

Thus, we now have:

$$H^i(SL_n\mathbb{Z}; \mathbb{Q}[SL_n\mathbb{Z}]) \underset{\substack{\uparrow \\ \overline{X}_n/SL_n \\ \text{is cpct}}}{=} H^i_c(\overline{X}_n; \mathbb{Q}) \overset{\substack{\text{Poincare duality} \\ \text{for } \overline{X}_n}}{=} \tilde{H}_{\binom{n+1}{2}-i-1}(\overline{X}_n, \partial\overline{X}_n; \mathbb{Q})$$

$$\overset{\substack{\text{LES of pair,} \\ \overline{X}_n \simeq \{*\}}}{=} \tilde{H}_{\binom{n+1}{2}-i-2}(\partial\overline{X}_n; \mathbb{Q})$$

$$\overset{\partial\overline{X}_n \simeq \mathcal{T}_n\mathbb{Q}}{=} \tilde{H}_{\binom{n+1}{2}-i-2}(\mathcal{T}_n\mathbb{Q}; \mathbb{Q})$$

Since $\mathcal{T}_n\mathbb{Q} \simeq \vee S^{n-2}$, H_* is concentrated in deg $n-2$ (and is free abelian for deg $n-2$)

i.e., precisely when:

$$\binom{n+1}{2} - i - 2 = n-2$$

$$\Leftrightarrow \boxed{i = \binom{n}{2}}$$

Conclusion :

$$H^i(SL_n\mathbb{Z}; \mathbb{Q}) \cong H_{\binom{n}{2}-i}(SL_n\mathbb{Z}; \tilde{H}_{n-2}(\mathcal{T}_n\mathbb{Q}; \mathbb{Q}))$$

$$\cong H_{\binom{n}{2}-i}(SL_n\mathbb{Z}; \underbrace{St_n}_{\text{Steinberg Module}} \otimes \mathbb{Q})$$