# COHOMOLOGY OF THE GRASSMANNIAN AND SYMMETRIC POLYNOMIALS

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Contents



In this expository paper we describe the cohomology ring of the complex Grassmannian, by establishing it as a quotient of the ring of symmetric polynomials. We do this by giving the Grassmannian a cell structure, and then using the duality of cup product with transversal intersections of cells, which can in turn be understood through linear algebra arguments.

This isomorphism allows us to borrow combinatorial tools, such as the Littlewood-Richardson rule, for multiplying symmetric polynomials to find the cup products of cohomology classes. This in turn helps to study transversal intersections of linear subspaces using duality, and so can be used to answer problems in enumerative geometry - although this application is not explored in this paper.

## 1. Definitions and Notation

We start with setting up notation.

**Definition.** The ***Complex Grassmannian*** $\boldsymbol{Gr}_r(\mathbb{C}^m)$ is the set of $r$-dimensional complex linear subspaces of $\mathbb{C}^m$. Let $n = m - r$.

We state without proof the following theorem:

**Theorem.** $\mathrm{Gr}_r(\mathbb{C}^m)$ is a compact closed (i.e. without boundary) complex manifold of (complex) dimension $rn$.

## 2. Cell Decomposition of the Grassmannian

Having a CW cell structure on a topological space is often a useful way to compute its (co)homologies. For Grassmannian manifolds, we have a ***Schubert cell decomposition***. To define this cell decomposition, we first need the notion of a *complete flag*.

**Definition.** A ***complete flag*** is an increasing sequence of linear subspaces

$$F_\bullet = \{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = \mathbb{C}^m$$

such that $\dim F_i = i$.

**Definition.** For a fixed flag $F_\bullet$ and partition $\lambda$, define the ***Schubert cell*** $\sigma_\lambda$ as follows:

$$\sigma_\lambda = \{V \in \mathrm{Gr}_r(\mathbb{C}^m) | \dim(V \cap F_{n+i-\lambda_i}) = i, \dim(V \cap F_{n+i-1-\lambda_i}) = i-1, 1 \leq i \leq r\}$$

**Remark.**
1. For any $V \in \mathrm{Gr}_r(\mathbb{C}^m)$ and $1 \leq i \leq r$, there has to be some $j_i$ so that $\dim(V \cap F_{j_i}) = i$. Thus what the definition above is saying is that $n+i-\lambda_i$ is the smallest number $j_i$ for which this holds.
2. Note that $n+i-\lambda_i < n+i+1-\lambda_{i+1}$ for all $i$.
3. Note that the above definition implies that $\sigma_\lambda$ is non-empty if and only if $\ell(\lambda) \leq r$ and $\lambda_1 \leq n$.

**Example.** Let $n = 3, r = 3$. Let $e_1, e_2, \ldots, e_6$ be the standard basis of $\mathbb{C}^6$, and take the standard flag given by $F_k = \langle e_1, e_2, \ldots, e_k \rangle$. Let $\lambda = (3, 1, 1)$. Note that $n+1-\lambda_1 = 3+1-3 = 1, n+2-\lambda_2 = 4, n+3-\lambda_3 = 5$. Take the 3-dimensional subspace $V$ of $\mathbb{C}^6$ spanned by $e_1, e_2+e_4, 2e_3+e_5$. Then $V \in \sigma_\lambda$.

**Example.** Take $n = 7, r = 5$. Let $e_1, e_2, \ldots, e_{12}$ be the standard basis of $\mathbb{C}^{12}$, and choose the standard flag as before. Let $\lambda = (5, 3, 2, 2, 1)$. Consider the 5 dimensional subspace $V$ of $\mathbb{C}^{12}$ spanned by $e_1+e_2+e_3, e_6, e_5+e_7+e_8, e_9, e_7+e_{11}$. Then $V \in \sigma_\lambda$.

**Remark.** The definition of $\sigma_\lambda$ above does depend on our choice of flag - however, as we will see later, the (co)homology classes determined by Schubert cells are in fact invariant under choice of flag. When we need to make a distinction between flags we'll denote a cell by $\sigma_\lambda(F)$. But for now we'll overlook this issue, and mostly work with the standard flag, defined in terms of the standard basis of $\mathbb{C}^m$.

**Proposition.** The $\sigma_\lambda$ give a cell decomposition of $\mathrm{Gr}_r(\mathbb{C}^m)$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition with $\lambda_1 \leq n$. The complex dimension of $\sigma_\lambda$ is $rn - |\lambda|$.

*Proof.* We'll show that $\sigma_\lambda$ is completely classified by the reduced row echelon form of its elements. For brevity, let $k_i = n+i-\lambda_i$.
Suppose $V \in \sigma_\lambda$. Since $\dim(V \cap F_{k_1}) = 1$, there must be a $v_1 \in V \cap F_{k_1}$. Furthermore, since $\dim(V \cap F_{k_1-1}) = 0$, $v_1$ must have a non-zero coordinate for $e_{k_1}$. We can normalise this to 1. Thus we can assume that

$$v_1 = (\underbrace{*, *, \ldots, *}_{k_1-1}, 1, 0, 0, \ldots, 0)$$

is a vector in $V$.
Similarly, since $\dim(V \cap F_{k_2}) = 2$, there must be a vector $v_2 \in V \cap F_{k_2}$, and we can further arrange for the $e_{k_2}$-coordinate of $v_2$ to be 1. Also, since we already have $v_1 \in V$ whose $e_{k_1}$-coordinate is 1, we can subtract an appropriate multiple of $v_1$ from $v_2$ to arrange for the $e_{k_1}$-coordinate of $v_2$ to be 0. Thus

$$v_2 = (\underbrace{*, \ldots, *}_{k_1-1}, 0, \underbrace{*, \ldots, *}_{k_2-k_1-1}, 1, 0, 0, \ldots, 0)$$

is in $V$. Similarly we can find a vector

$$v_3 = (\underbrace{*, \dots, *}_{k_1-1}, 0, \underbrace{*, \dots, *}_{k_2-k_1-1}, 0, \underbrace{*, \dots, *}_{k_3-k_2-1}, 1, 0, 0, \dots, 0)$$

in $V$, and so on.

If we arrange these vectors to form the rows of a matrix, we obtain the row reduced echelon form associated to $V$, as follows:

$$\begin{bmatrix} \overbrace{*\cdots*}^{k_1-1} & 1 & \overbrace{0\dots0}^{k_2-k_1-1} & 0 & \overbrace{0\dots0}^{k_3-k_2-1} & 0 & 0 & 0 & \dots & 0 \\ *\cdots* & 0 & *\cdots* & 1 & 0\dots0 & 0 & 0 & 0 & \dots & 0 \\ *\cdots* & 0 & *\cdots* & 0 & *\cdots* & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

It's not hard to see that a linear subspace $V$ is in $\sigma_\lambda$ if and only if its reduced row echelon form has the form given above. So this shows that $\sigma_\lambda$ is homeomorphic to $\mathbb{C}^N$, where $N$ is the number of $*$ in the matrix above. We can count $N$ as follows:

$$\begin{aligned} N &= r(k_1-1) + (r-1)(k_2-k_1-1) + (r-2)(k_3-k_2-1) + \dots + (r-i)(k_{i+1}-k_i-1) + \dots \\ &= r(n-\lambda_1) + (r-1)(\lambda_1-\lambda_2) + (r-2)(\lambda_2-\lambda_3) + \dots + (r-i)(\lambda_i-\lambda_{i+1}) + \dots \\ &= rn - \lambda_1 - \lambda_2 - \lambda_3 - \dots - \lambda_i - \dots \\ &= rn - |\lambda| \end{aligned}$$

Finally, note that given a row reduced echelon matrix, we can recover the corresponding partition $\lambda$. By the uniqueness of row reduced echelon form, it follows that each open cell $\sigma_\lambda(\cong \mathbb{C}^{rn-|\lambda|})$ maps homeomorphically onto the Grassmannian. □

**Remark.** The matrix constructed above is really obtained from a reduced row echelon matrix after reversing the entries in each row from left to right and each column from top to bottom, and so is not, strictly speaking, a reduced row echelon matrix itself. However since we can still use the same existence and uniqueness result for it, we'll overlook this subtlety here. (True reduced row echelon matrices will appear later though, when we define 'reverse flags')

We wrap up this section with a description of the closure of a Schubert cell.

**Proposition.** For a partition $\lambda$, the closure of the cell $\sigma_\lambda$ is given as follows:

$$\overline{\sigma_\lambda} = \{V \in Gr_r(\mathbb{C}^m) | \dim(V \cap F_{n+i-\lambda_i}) \geq i, 1 \leq i \leq r\}$$

*Proof.* Clearly, the left hand side is contained in the right hand side. We can also see that the right hand side is closed, as follows:

Note that the right hand side is the intersection of $E_i = \{V \in \mathrm{Gr}_r(\mathbb{C}^m) | \dim(V \cap F_{n+i-\lambda_i}) \geq i\}$ for a fixed $i$, as $i$ ranges from 1 to $r$. Thus it's enough to show that each of these individual sets $E_i$ is closed. Take $V \in E_i$. Write out $V$ as an $r \times m$ matrix by taking a basis for $V$ and arranging them in the rows of the matrix (by writing each basis element in terms of the standard basis $e_1, e_2, \dots, e_m$). The linear subspace $F_{n+i-\lambda_i}$ corresponds to the first $n+i-\lambda_i$ columns of the matrix, so the condition that $\dim(V \cap F_{n+i-\lambda_i}) \geq i$ is equivalent to requiring that in the reduced row ecehelon form of this matrix, that atleast $i$ rows should have all 0's in their last $m-(n+i-\lambda_i)=r-i+\lambda_i$ columns. This is equivalent to saying that if we take our original matrix and consider only its last $r-i+\lambda_i$ columns, then this truncated matrix should have rank $\leq r-i-1$. This further translates to saying that all $(r-i)\times(r-i)$ minors of the truncated matrix should vanish. This implies that $E_i$ is closed.

So we now need to show that $\overline{\sigma_\lambda} \supset \{V | \dim(V \cap F_{n+i-\lambda_i}) \geq i, 1 \leq i \leq r\}$. For this it is enough to take a $V$ in the right hand side and find a sequence of elements in $\sigma_\lambda$ that converge to $V$.

We know that $V \in \sigma_\mu$ for some unique $\mu$. The condition that $\dim(V \cap F_{n+i-\lambda_i}) \geq i$ says that the pivots in the reduced row echelon form of $V$ appear before $n+i-\lambda_i$ on the $i$th row. i.e, $n+i-\mu_i \leq n+i-\lambda_i$ or equivalently, $\lambda_i \leq \mu_i$ for all $i$. Conversely if $V \in \sigma_\mu$ with $\lambda_i \leq \mu_i$ for all $i$ then $\dim(V \cap F_{n+i-\lambda_i}) \geq i$ for all $i$.
First we'll show that for a fixed $j$, if $\mu$ satisfies $\mu_i = \lambda_i$ for all $i \neq j$ and $\mu_j > \lambda_j$, then $V \in \overline{\sigma_\lambda}$. In this case, the $j$th row of the reduced row echelon form of $V$ looks like:

$$\begin{bmatrix} t_1 & t_2 & \dots & t_{n+j-\mu_j-1} & 1 & 0 & 0 & \dots & 0 & 0\dots \end{bmatrix}$$

Take a sequence $\{V_k\}$ in $\sigma_\lambda$ as follows: All but the $j$th row of $V_k$ agree with $V$, and the $j$th row of $V_k$ is given by:

$$\begin{bmatrix} t_1 & t_2 & \dots & t_{n+j-\mu_j-1} & 1 & \frac{1}{k} & \frac{1}{k} & \dots & \frac{1}{k} & 0\dots \end{bmatrix},$$

with the last $\frac{1}{k}$ appearing in the $(n+j-\lambda_j)$th position. Since the $(n+j-\lambda_j)$th entry is non-zero, $V_k$ is in $\sigma_\lambda$, and clearly $V_k \to V$ as $k \to \infty$.
Now for the proof in the general case, we proceed as follows: Given $V \in \sigma_\mu$, first find a sequence in $\sigma_{(\lambda_1,\mu_2,\dots,\mu_r)}$ that converges to $V$ using the previous case. Then for each chosen element in $\sigma_{(\lambda_1,\mu_2,\dots,\mu_r)}$, find a sequence in $\sigma_{(\lambda_1,\lambda_2,\mu_3,\dots,\mu_r)}$ converging to it, and so on. This will show that $\sigma_\mu \subset \overline{\sigma_{(\lambda_1,\mu_2,\dots,\mu_r)}} \subset \overline{\sigma_{(\lambda_1,\lambda_2,\mu_3,\dots,\mu_r)}} \subset \dots \subset \overline{\sigma_\lambda}$, and in particular, $\sigma_\mu \subset \overline{\sigma_\lambda}$, as desired. ☐

## 3. Poincare Duality and Cup Product

In this short section we state two results from algebraic topology that we will be using to compute $H^*(Gr_r(\mathbb{C}^m))$. Throughout this paper we'll be working with coefficients in $\mathbb{Z}$, so we may at times omit this from the notation.

**Poincare Duality.**

**Theorem.** If $M$ is a compact, closed oriented manifold of (real) dimension $n$, then there is a canonical isomorphism

$$H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$$

For a homology class $[A] \in H_{n-k}(M)$, we denote its dual class in $H^k(M)$ by $[A]^*$.

**Remark.** Since $\mathrm{Gr}_r(\mathbb{C}^m)$ is a complex manifold, in particular it is orientable and so Poincare Duality applies.

**Cup Product and Intersection.** The next result states that the cup product in the cohomology ring of $M$ is dual to intersecting homology classes.

**Theorem.** If $M$ is a compact closed oriented manifold of dimension $n$, and $A, B$ are oriented submanifolds of $M$ that intersect transversely, with $[A], [B]$ denoting their respective homology classes, then

$$[A]^* \smile [B]^* \cong [A \cap B]^*$$

## 4. Cohomology of the Grassmannian

**Additive Cohomology Structure.** We can use the cell decomposition of $\mathrm{Gr}_r(\mathbb{C}^m)$ to determine the additive structure of $H^*(\mathrm{Gr}_r(\mathbb{C}^m))$, as follows: Since all cells have even (real) dimension, all the boundary maps in the chain complex so obtained must be zero. This implies that all the homology groups of $\mathrm{Gr}_r(\mathbb{C}^m)$ are free abelian groups - in odd dimensions the homology is trivial, and in an even dimension $2k$ we have a free abelian basis formed by homology classes $[\overline{\sigma_\lambda}]$, where $\lambda$ is a partition such that $\lambda_1 \leq n$, $\ell(\lambda) \leq r$, and $k = rn - |\lambda|$. By Poincare duality, knowing the degree-$2k$ homology means we also know the degree-($2rn - 2k$) cohomology. The cohomology classes $[\overline{\sigma_\lambda}]^*$ generating the cohomology groups are called ***Schubert cycles***.
We summarise this in the following theorem:

**Theorem.** $H^{2k+1}(\mathrm{Gr}_r(\mathbb{C}^m); \mathbb{Z}) = 0$ for all $k = 0, 1, 2, \ldots$. For even dimensions, the set of Schubert cycles $[\overline{\sigma_\lambda}]^*$ such that $\lambda_1 \leq n, \ell(\lambda) \leq r,$ and $|\lambda| = k$, form a basis for $H^{2k}(\mathrm{Gr}_r(\mathbb{C}^m); \mathbb{Z})$.

**Towards Ring Structure.** We are now interested in analysing the multiplicative structure of the cohomology ring of $Gr_r(\mathbb{C}^m)$. The remainder of this and the next section will be devoted to proving the following result:

**Theorem.** As a ring, we have

$$H^*(\mathrm{Gr}_r(\mathbb{C}^m); \mathbb{Z}) \cong \Lambda_r/(h_{m-r+1}, h_{m-r+2}, \ldots) = \Lambda_r/(h_{n+1}, h_{n+2}, \ldots)$$

where $\Lambda_r$ denotes ring of symmetric polynomials in $x_1, x_2, \ldots, x_r$ (with integer coefficients), and the $h_k$ denote the homogeneous symmetric functions.
Moreover, this isomorphism is given by $s_\lambda \mapsto [\sigma_\lambda]^*$, where $s_\lambda$ denotes the Schur polynomial.

In the next subsection we present an outline of a proof this theorem.

**Outline of Proof.** From now on, we will (severely) abuse notation and use $\sigma_\lambda$ to denote both the Schubert cell and its corresponding Schubert cycle $[\overline{\sigma_\lambda}]^*$.
Recall **Pieri's rule** for symmetric functions:

**Theorem.**

$$s_\lambda \cdot s_{(k)} = \sum_{\lambda \uparrow \lambda', |\lambda'| = |\lambda| + k} s_{\lambda'}$$

We will show that Pieri's rule also holds for the cohomology classes $\sigma_\lambda$ in $H^*(\mathrm{Gr}_r(\mathbb{C}^m))$, i.e. that

$$\sigma_\lambda \cdot \sigma_{(k)} = \sum_{\lambda \uparrow \lambda', |\lambda'| = |\lambda| + k} \sigma_{\lambda'}$$

This will require a fair bit of work, and is postponed to the next section.

Assuming Pieri's rule holds, we can proceed as follows: Note that $h_k = s_k$, and that $\Lambda_k$ is freely generated as a ring in $h_1, h_2, \ldots$. Thus we can uniquely describe a ring homomorphism $\Lambda_k \to H^*(\mathrm{Gr}_r(\mathbb{C}^m))$ that sends $h_k \mapsto \sigma_{(k)}$. We can then use Pieri's rule, that holds in both

$\Lambda_k$ and $H^*(Gr_r(\mathbb{C}^m))$, to show that this ring map sends $s_\lambda \mapsto \sigma_\lambda$.
This will imply that the ring map is surjective since all generators $\sigma_\lambda$ of $H^*(\mathrm{Gr}_r(\mathbb{C}^m))$ are in the image; Since $\sigma_{(k)} = 0$ for $k > n$ and $\neq 0$ otherwise, and because the Schur polynomials $s_\lambda$ are linearly independent, it will follow that the kernel of this map is generated by $s_{n+1}, s_{n+2}, \dots$ (which are the same as $(h_{n+1}, h_{n+2}, \dots)$), and so the isomorphism will follow.

Now let's see why Pieri's rule implies that $s_\lambda \mapsto \sigma_\lambda$. We already know that $s_{(k)} \mapsto \sigma_{(k)}$, so we know the result holds when $\ell(\lambda) = 1$. We will induct on the length of $\lambda$.

Suppose the result holds for partitions of length $l$, and we want to show it for length $l+1$. Take a partition $(\lambda_1, \lambda_2, \dots, \lambda_l)$. By Pieri's rule, we have

$$s_\lambda \cdot h_1 = \sum (s \text{ indexed by partitions with } l \text{ parts}) + s_{(\lambda_1, \lambda_1, \dots, \lambda_l, 1)}$$

A similar equation holds for the $\sigma_\mu$'s, and since all the terms in the above equation except for $s_{(\lambda_1, \dots, \lambda_l, 1)}$ map to their $\sigma$-counterparts, it follows that $s_{(\lambda_1, \dots, \lambda_l, 1)} \mapsto \sigma_{(\lambda_1, \dots, \lambda_l, 1)}$. Thus we have proved the result holds for $(l+1)$-length partitions whose last part is 1.

We shall now successively increase the size of this $(l+1)$-th part, by multiplying $s_\lambda$ with $h_k$. This will involve another induction step. So suppose we have proved the result for $(l+1)$-length partitions whose last part is $\leq k$ (so for what we showed above is the case $k = 1$). We can increase the size of this last part to $k+1$ as follows:

By Pieri's rule, we have

$$s_{(\lambda_1, \lambda_2, \dots, \lambda_l)} \cdot h_{(k+1)} = \sum (s \text{ indexed by partitions with } \leq l{+}1 \text{ parts, with } (l{+}1)\text{-th part} \leq k) + s_{(\lambda_1, \dots, \lambda_l, k+1)}$$

A similar result holds for the $\sigma$'s, and as before, since all but one of the above terms map to their $\sigma$-counterparts, this will prove that $s_{(\lambda_1, \dots, \lambda_l, k+1)} \mapsto \sigma_{(\lambda_1, \dots, \lambda_l, k+1)}$.

## 5. Proof of Pieri's Rule

In this section we prove Pieri's rule for the Schubert cycles $\sigma_\lambda$. Since we want to prove a formula involving cup products in cohomology, and cup product is dual to tranversal intersections, we will analyse intersections of Schubert cells. In general, two cells $\overline{\sigma_\lambda}$ and $\overline{\sigma_{\lambda'}}$ (defined using the same flag) do not intersect transversely, so it will be convenient to use different flags for different cells. The use of this is justified by the following theorem, which says that the (co)homology class defined by $\overline{\sigma_\lambda}$ is independent of the flag used to define it. For clarity, let's denote the Schubert cell defined by partition $\lambda$ and flag $F$ by $\sigma_\lambda(F)$.

**Theorem. (Flag Invariance)** Let $F, F'$ be two complete flags. Then $[\overline{\sigma_\lambda(F)}]^*$ and $[\overline{\sigma_\lambda(F')}]^*$ represent the same classes in cohomology. The same holds for homology.

*Proof.* Note that the multiplication map $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ gives a transitive action of $\mathrm{GL}_m(\mathbb{C})$ on complete flags, so we can find a $g \in \mathrm{GL}_m(\mathbb{C})$ such that $g(F_i) = F'_i$ for all $i$. This action of $\mathrm{GL}_m(\mathbb{C})$ is continuous, and since $\mathrm{GL}_m(\mathbb{C})$ is connected, there is a path $P : [0,1] \to \mathrm{GL}_m(\mathbb{C})$ with $P(0) = g$ and $P(1) = \mathrm{Id}_m$. This path gives a homotopy between multiplication by $g : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ and $\mathrm{Id} : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$. Thus the induced maps $g^*, \mathrm{Id}^*$ on cohomology are the same. This implies that $[\overline{\sigma_\lambda(F)}]$ and $[\overline{\sigma_\lambda(F')}]$ represent the same class in cohomology. Similar reasoning gives the analogous proof for homology. ☐

So this theorem justifies our use of $\sigma_\lambda$ to denote cohomology classes, without reference to the flag used to define the Schubert cell.

One particular kind of flag we'd like to work with is the ***reverse flag***, defined as follows:

**Definition.** Given a complete flag $F_\bullet$, such that $e_1, e_2, \ldots, e_m$ is a basis of $\mathbb{C}^m$ with $F_i = \langle e_1, e_2, \ldots, e_i \rangle$, it's ***reverse flag*** $\tilde{F}_\bullet$ is defined by $\tilde{F}_i = \langle e_m, e_{m-1}, \ldots, e_{m-i+1} \rangle$.

**Example.** Let $n = r = 3$, and $\lambda = (3, 1, 1)$. Let $F$ be the standard flag of $\mathbb{C}^6$, and $\tilde{F}$ the corresponding reverse flag. We saw earlier that $n + 1 - \lambda_1 = 3 + 1 - 3 = 1, n + 2 - \lambda_2 = 4, n + 3 - \lambda_3 = 5$. So an element in $\sigma_\lambda(F)$ looks like:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 1 & 0 & 0 \\ 0 & * & * & 0 & 1 & 0 \end{bmatrix}$$

If we took an element of $\sigma_\lambda(\tilde{F})$, and again wrote it out as a matrix whose rows are basis vectors for the space (with the basis vectors written down in terms of the standard basis), then that matrix in general would look like:

$$\begin{bmatrix} 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that this matrix was obtained from the previous matrix by reversing its rows and columns (the latter matrix is in fact in standard reduced row echelon form). In general, given $\lambda$ and $F$, we obtain the general matrix for $\sigma_\lambda(\tilde{F})$ by taking the general matrix for $\sigma_\lambda(F)$ and reversing its rows and columns.

We now work towards understanding cup products of Schubert cycles.

**Theorem. (Duality Theorem)** For two partitions $\lambda, \mu$, we have:

$$\sigma_\lambda \smile \sigma_\mu = \begin{cases} 1 & \lambda_i + \mu_{r+1-i} = n \text{ for all } 1 \leq i \leq r \\ 0 & \lambda_i + \mu_{r+1-i} > n \text{ for any } i \end{cases}$$

*Proof.* Since cup product is dual to intersection, we'll prove this by looking at the intersection of the Schubert cells $\overline{\sigma_\lambda(F)}$ and $\overline{\sigma_\mu(\tilde{F})}$, where $F$ is the standard flag. If we use our method of viewing elements in these cells as matrices, we know that the $i$th row of the matrix for $\overline{\sigma_\lambda(F)}$ has a 1 to the left or at the $n + i - \lambda_i$ position, and 0's to the right of it. The $i$th row of something in $\overline{\sigma_\mu(\tilde{F})}$ has a 1 to the right or at the $m - (n + (r + 1 - i) - \mu_{r+1-i}) + 1$ position (the change in indices corresponds to having reversed columns and rows), and 0's to the left of it. Thus for the two cells to have any intersection, the position of the 1's in $\sigma_\lambda(F)$ can't be on the left of the 1's in the corresponding rows of the $\sigma_\mu(F')$. That is, if we have

$$n+i-\lambda_i < m-(n+(r+1-i)-\mu_{r+1-i})+1 \iff n+i-\lambda_i < i+\mu_{r+1-i} \iff \lambda_i+\mu_{r+1-i} > n$$

for any $i$, then the intersection has to be empty.

If $\lambda_i + \mu_{r+1-i} = n$ for all $i$, then this means that the 1's are all in the same positions on each row, and the rest of the entries of the matrix are 0. Thus the two cells intersect in exactly one point. □

**Definition.** If $\lambda, \mu$ satisfy the condition above, they are called ***dual partitions***, with $\mu$ (resp, $\lambda$) denoted as $\tilde{\lambda}$ (resp, $\tilde{\mu}$).

Here's how the duality theorem will be useful - Since the product $\sigma_\lambda \smile \sigma_{(k)}$ will have dimension $|\lambda| + k$ in the cohomology ring, we know that:

$$\sigma_\lambda \smile \sigma_{(k)} = \sum_{|\lambda'|=|\lambda|+k} c_{\lambda'}\sigma_{\lambda'}$$

for some integers $c_{\lambda'}$. We want to show that $c_{\lambda'} = 1$ if and only if $\lambda \uparrow \lambda'$, and zero otherwise. For this we will use the duality theorem. For a fixed $\mu$, multiplying the above equation with $\sigma_{\tilde{\mu}}$ gives us

$$\sigma_{\tilde{\mu}} \smile \sigma_\lambda \smile \sigma_{(k)} = \sum_{\lambda'} c_{\lambda'}(\sigma_{\tilde{\mu}} \smile \sigma_{\lambda'}) = c_\mu$$

Thus we need to show that $\sigma_{\tilde{\mu}} \smile \sigma_\lambda \smile \sigma_{(k)} = 1$ if and only if $\lambda \uparrow \mu$ and $|\mu| = |\lambda| + k$, and 0 otherwise. As before, we will study this product of cohomology classes using intersections of Schubert cells, with suitably chosen flags.

For brevity and consistency with notation, we replace $\tilde{\mu}$ in the above statement with $\mu$. So what we want to show is: if $|\tilde{\mu}| = |\lambda| + k$, then $\sigma_\mu \smile \sigma_\lambda \smile \sigma_{(k)} = 1$ if and only if $\lambda \uparrow \tilde{\mu}$, and 0 otherwise.

If $|\tilde{\mu}| = |\lambda| + k$, then the condition that $\lambda \uparrow \tilde{\mu}$ is equivalent to the following inequalities:

$$n - \lambda_r \geq \mu_1 \geq n - \lambda_{r-1} \geq \cdots \geq n - \lambda_1 \geq \mu_r \geq 0 \tag{1}$$

Our first goal will be to analyse the cup product $\sigma_\mu \smile \sigma_\lambda$. As stated before, we'll do this by analysing intersection of Schubert cells, using the standard flag $F$ for $\lambda$ and its reverse flag $\tilde{F}$ for $\mu$. From the proof of the duality theorem, we know that for this intersection to be non-empty, we must have $\lambda_i + \mu_{r+1-i} \leq n$ for all $1 \leq i \leq r$. We also saw that if we write down an element of $\overline{\sigma_\lambda(F)}$ as a matrix, then the nonzero entries of row $i$ occur to the left or at position $n + i - \lambda_i$, and for $\overline{\sigma_\mu(\tilde{F})}$ they occur to the right or at the position $m - (n + (r + 1 - i) - \mu_{r+1-i}) + 1 = i + \mu_{r+1-i}$. Thus for a matrix in the intersection of these two cells, all its non-zero entries in row $i$ must fall between positions $n + i - \lambda_i$ and $i + \mu_{r+1-i}$.

**Example.** Take $n = r = 3, \lambda = (3, 1, 1)$. Let $k = 1, \mu = (2, 1, 0)$. Then the general form for an element in $\overline{\sigma_\lambda(F)}$ is:

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

and the general form for an element in $\overline{\sigma_\mu(\tilde{F})}$ is:

$$\begin{bmatrix} * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

and so an element in the intersection looks like:

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

We fix some notation before continuing further analysis:

**Definition.** (1) Let $C_i$ denote the rowspan of row $i$ of the above described intersection. Thus $C_i$ is the span of all $e_j$ such that

$$i + \mu_{r+1-i} \leq j \leq n + i - \lambda_i$$

Let $C = C_1 + C_2 + \cdots + C_r$.

(2) Let $A_i = F_{n+i-\lambda_i} = \text{Span}(e_1, e_2, \ldots, e_{n+i-\lambda_i})$ for $1 \leq i \leq r$. Set $A_0 = 0$.

(3) Let $B_i = \tilde{F}_{n+i-\mu_i} = \text{Span}(e_m, e_{m-1}, \ldots, e_{m-(n+i-\mu_i)+1}) = \text{Span}(e_m, e_{m-1}, \ldots, e_{r+1-i+\mu_i})$ for $1 \leq i \leq r$. Set $B_0 = 0$.

Thus note that $C_i = A_i \cap B_{r+1-i}$.

The rest of our analysis of $\sigma_\mu \smile \sigma_\lambda$ involves linear algebra arguments with the $A_i, B_i, C_i$, which are broken down in several steps below.

**Lemma.** (1) $\sum_{i=1}^{r} \dim C_i = r + k$.

(2) $C = C_1 + C_2 + \cdots + C_r$ is a direct sum if and only if the inequalities 1 hold.

*Proof.* (1) From our description of $C_i$, it is clear that

$$\dim C_i = (n + i - \lambda_i) - (i + \mu_{r+1-i}) + 1 = n + 1 - \lambda_i - \mu_{r+1-i}$$

Thus

$$\sum_{i=1}^{r} \dim C_i = \sum_{i=1}^{r} (n+1-\lambda_i - \mu_{r+1-i}) = rn + r - |\lambda| - |\mu| = rn + r - |\lambda| - (rn - |\lambda| - k) = r + k,$$

as desired.

(2) This is again clear from our description of $C_i$. Since $C_i$ is spanned by all $e_j$ such that $i + \mu_{r+1-i} \leq j \leq n + i - \lambda_i$, the only way for the $C_i$ to be linearly independent is if

$$i + \mu_{r+1-i} \leq n + i - \lambda_i < (i+1) + \mu_{r-i}$$

for all $i$, which reduces to the inequalities

$$\mu_{r+1-i} \leq n - \lambda_i \leq \mu_{r-i}$$

for all $i$, which is precisely what 1 says.

□

**Lemma.** $C = \bigcap_{i=0}^{r} (A_i + B_{r-i})$.

*Proof.* First note that $A_0 \subset A_1 \subset \cdots \subset A_r$ and $B_0 \subset B_1 \subset \cdots \subset B_r$. We shall use the fact that for linear subspaces $W, X, Y \subset \mathbb{C}^m$ with $W \subset Y$, we have $(W + X) \cap Y = W + (X \cap Y)$. Applying this twice to the case where $W \subset Y, Z \subset X$, we get $(W + X) \cap (Y + Z) =$

$W + (X \cap (Y + Z)) = W + (X \cap Y) + Z$. Applying this $r$ times gives us:

$$\begin{aligned}
\bigcap_{i=0}^{r}(A_i + B_{r-i}) &= (A_0 + B_r) \cap (A_1 + B_{r-1}) \cap \cdots \cap (A_r + B_0) \\
&= (A_0 + A_1 \cap B_r + B_{r-1}) \cap (A_2 + B_{r-2}) \cap \cdots \cap (A_r + B_0) \\
&= (A_0 + A_1 \cap B_r + A_2 \cap B_{r-1} + B_{r-2}) \cap (A_3 + B_{r-3}) \cap \cdots \cap (A_r + B_0) \\
&= \dots \\
&= A_0 + A_1 \cap B_r + A_2 \cap B_{r-1} + \cdots + A_r \cap B_1 + B_0 \\
&= \sum_{i=1}^{r} A_i \cap B_{r+1-i} = \sum_{i=1}^{r} C_i = C,
\end{aligned}$$

as desired.

□

**Lemma.** (1) If $V \in \overline{\sigma_\lambda(F)} \cap \overline{\sigma_\mu(\tilde{F})}$, then $V \subset C$.

(2) If in addition $C_1, C_2, \dots, C_r$ are linearly independent, then $\dim(V \cap C_i) = 1$ for all $i$, and $V = (V \cap C_1) \oplus (V \cap C_2) \oplus \cdots \oplus (V \cap C_r)$.

*Proof.* (1) By the previous lemma, it is enough to show that $V \subset A_i + B_{r-i}$ for all $i$. First suppose that $A_i \cap B_{r-i} = 0$. $V \in \overline{\sigma_\lambda(F)}$ implies that $\dim(V \cap A_i) \geq i$, and $V \in \overline{\sigma_\mu(\tilde{F})}$ implies that $\dim(V \cap B_{r-i}) \geq r - i$. Since $\dim V = r$ and $A_i \cap B_{r-i} = 0$, this implies that $\dim(V \cap (A_i + B_{r-i})) = r$, i.e. $V \subset A_i + B_{r-i}$.

If $A_i \cap B_{r-i} \neq 0$, then this means that there is some $e_j$ such that $m - i - \mu_{r-i} \leq j \leq n + i - \lambda_i$, and so $m - i - \mu_{r-i} \leq n + i - \lambda_i$. Since $A_i$ is spanned by all $e_j$ with $j \leq n + i - \lambda_i$ and $B_{r-i}$ is spanned by those with $m - i - \mu_{r-i} \leq j$, it follows that $A_i + B_{r-i} = \mathbb{C}^m$. Then certainly $V \subset A_i + B_{r-i}$.

(2) We know in this case that $C_i = A_i \cap B_{r+1-i}$ and so, following a similar reasoning as in the proof of part(1), we have $A_i + B_{r+1-i} = \mathbb{C}^m$. Thus, $(V \cap A_i) + (V \cap B_{r-i+1}) = V$, and $(V \cap A_i) \cap (V \cap B_{r+1-i}) = V \cap C_i$. Now, the condition on $V$ implies that $\dim(V \cap A_i) \geq i$ and $\dim(V \cap B_{r+1-i}) \geq r + 1 - i$, and by a dimension count we have $\dim(V \cap C_i) \geq (i) + (r + 1 - i) - (r) = 1$.

If the $C_i$ are linearly independent, then $V$ contains the direct sum of the $V \cap C_i$. Since each of these has dimension $\geq 1$ and $V$ itself has dimension $r$, it follows that $\dim(V \cap C_i) = 1$ for all $i$ and that $V$ must be exactly equal to the direct sum of the $V \cap C_i$.

□

We eventually want to understand the cup product $\sigma_\mu \smile \sigma_\lambda \smile \sigma_{(k)}$. We need to choose a flag $L_\bullet$ using which we'll intersect the Schubert cell $\overline{\sigma_{(k)}(L_\bullet)}$ with $\overline{\sigma_\mu(\tilde{F})} \cap \overline{\sigma_\lambda(F)}$.
Note that since the partition $(k)$ has length 1, we have

$$\overline{\sigma_{(k)}(L)} = \{X \in \mathrm{Gr}_r(\mathbb{C}^m) : \dim(X \cap L_{n+1-k}) \geq 1\},$$

so this cell only depends on the $n + 1 - k$ dimensional subspace $L_{n+1-k}$. For brevity let's replace $L_{n+1-k}$ with $L$, and define $\overline{\sigma(L)} := \{X : \dim(X \cap L) \geq 1\}$. We'll pick the subspace

$L$ later, as per what makes our argument convenient.

We want to show that $\overline{\sigma_\mu(\tilde{F})} \cap \overline{\sigma_\lambda(F)} \cap \overline{\sigma(L)}$ has exactly one element when the inequalities 1 hold, and an empty intersection otherwise.
If 1 fails to hold, then we know that $C = C_1 + C_2 + \cdots + C_r$ is not a direct sum, and so $\dim C \leq (\sum_{i=1}^r \dim C_i) - 1 = r + k - 1$. Therefore a generic $n+1-k$-dimensional subspace $L$ will only intersect $C$ at the origin. We know that any $V \in \overline{\sigma_\mu(\tilde{F})} \cap \overline{\sigma_\lambda(F)}$ is contained in $C$, so $V$ will only intersect $L$ at the origin. Thus $\dim(V \cap L) = 0$, and so $V \notin \overline{\sigma(L)}$. Thus the intersection of the three Schubert cells is empty.

If 1 does hold, then we know that $C = C_1 \oplus C_2 \oplus \cdots \oplus C_r$ has dimension $r + k$, and so a generic $L$ intersects $C$ in a line of the form $\mathbb{C} \cdot v$, where $v = u_1 \oplus u_2 \oplus \cdots \oplus u_r$, $u_i \in C_i$. Since we have freedom over choosing $L$, we can also assume that $u_i \neq 0$. The condition that $\dim(V \cap L) \geq 1$ then forces $V$ to contain $v$. Further, since $V = V \cap C_1 \oplus V \cap C_2 \oplus \cdots \oplus V \cap C_r$, this implies that $u_i \in V$, and so $V = \langle u_1, u_2, \ldots, u_r \rangle$. Thus, the three Schubert cells meet only at the point $\langle u_1, u_2, \ldots, u_r \rangle$. This completes the proof.

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